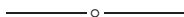


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Short Division of Polynomials

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When I teach, I always tell my students to ask me “why”. So when I was teaching synthetic division in precalculus, my student Heysel Marte promptly asked why the algorithm worked. I told her that it was an excellent question, thought for a moment, and then presented the following sketch to the class.

“Suppose that we want to divide $ax^3 + bx^2 + cx + d$ by $x - k$. Then we are seeking an answer of the form

$$ax^3 + bx^2 + cx + d = (ex^2 + fx + g)(x - k) + r.$$

Expanding the right-hand side we get $ex^3 + (f - ek)x^2 + (g - fk)x + r - gk$. Comparing coefficients we see that $e = a$, $f = b + ek$, $g = c + fk$, and $r = d + gk$, which is exactly what synthetic division is doing.”

$$\begin{array}{r|cccc} k & a & b & c & d \\ & & ek & fk & gk \\ \hline & e & f & g & r \end{array}$$

As a casual comment, I pointed out the caution in the textbook about the limitations of the algorithm, but after the class, I asked myself why the idea above could not be extended to divisors of higher degrees. Then I realized that it COULD!

Suppose that we want to divide $ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$ by $x^2 - kx - l$. Then we are seeking an answer of the form

$$\begin{aligned} ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g \\ = (mx^4 + nx^3 + ox^2 + px + q)(x^2 - kx - l) + (rx + s). \end{aligned}$$

Expanding the right-hand side and comparing coefficients we see that $m = a$, $n = b + mk$, $o = c + nk + ml$, $p = d + ok + nl$, $q = e + pk + ol$, $r = f + qk + pl$, and $s = g + ql$. We now display these relations in a format of synthetic division.

$$\begin{array}{r|cccccc} k & l & a & b & c & d & e & f & g \\ & & & mk & nk & ok & pk & qk & \\ & & & & ml & nl & ol & pl & ql \\ \hline & & m & n & o & p & q & r & s \end{array}$$

This display clearly suggests an algorithm: Drop down a as m . Multiply m with k and l and write the answers in the next two columns diagonally. Add the second column to get n . Multiply n with k and l and write the answers in the next two columns diagonally. Add the third column to get o , and so on.

However, we quickly realize that when the degree of the divisor exceeds half of the degree of the dividend, this diagonal form is not space-efficient, as shown below when the divisor is changed to $x^5 - hx^4 - ix^3 - jx^2 - kx - l$.

Diagonal Form

h	i	j	k	l	a	b	c	d	e	f	g
						mh	nh				
							mi	ni			
								mj	nj		
									mk	nk	
										ml	nl
					m	n	o	p	q	r	s

We can make this notation more compact by writing all the successive products horizontally.

Horizontal Form

h	i	j	k	l	a	b	c	d	e	f	g
						mh	mi	mj	mk	ml	
							nh	ni	nj	nk	nl
					m	n	o	p	q	r	s

Comparing the two forms, we notice that it really doesn't matter which row each product is written in, as long as it is in the correct column. Therefore, a space-efficient form should employ a 'compact' strategy: *In each column, write the product in the earliest available row.* With this strategy, we can carry out both divisions above in two rows without having to switch forms.

Compact Form

k	l	a	b	c	d	e	f	g
			mk	ml	nl	ol	pl	ql
				nk	ok	pk	qk	
		m	n	o	p	q	r	s

Compact Form

h	i	j	k	l	a	b	c	d	e	f	g
						mh	mi	mj	mk	ml	nl
							nh	ni	nj	nk	
					m	n	o	p	q	r	s

Still, we would need to know in advance how many rows are needed between the starting row and the answer row. We avoid this issue by writing the products above the starting line, converting the compact strategy into a natural 'piling by gravity.' In an attempt to undermine the dominance of long division of polynomials in school

