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Short Division of Polynomials

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When I teach, I always tell my students to ask me "why". So when I was teaching synthetic division in precalculus, my student Heysel Marte promptly asked why the algorithm worked. I told her that it was an excellent question, thought for a moment, and then presented the following sketch to the class.

"Suppose that we want to divide $ax^3 + bx^2 + cx + d$ by $x - k$. Then we are seeking an answer of the form

$$ax^3 + bx^2 + cx + d = (ex^2 + fx + g)(x - k) + r.$$

Expanding the right-hand side we get $ex^3 + (f - ek)x^2 + (g - fk)x + r - gk$. Comparing coefficients we see that $e = a$, $f = b + ek$, $g = c + fk$, and $r = d + gk$, which is exactly what synthetic division is doing."

$$\begin{array}{r|rrr|r} k & a & b & c & d \\ & & ek & fk & gk \\ \hline & e & f & g & r \end{array}$$

As a casual comment, I pointed out the caution in the textbook about the limitations of the algorithm, but after the class, I asked myself why the idea above could not be extended to divisors of higher degrees. Then I realized that it COULD!

Suppose that we want to divide $ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$ by $x^2 - kx - l$. Then we are seeking an answer of the form

$$\begin{aligned} &ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g \\ &\qquad = (mx^4 + nx^3 + ox^2 + px + q)(x^2 - kx - l) + (rx + s). \end{aligned}$$

Expanding the right-hand side and comparing coefficients we see that $m = a$, $n = b + mk$, $o = c + nk + ml$, $p = d + ok + nl$, $q = e + pk + ol$, $r = f + qk + pl$, and $s = g + ql$. We now display these relations in a format of synthetic division.

$$\begin{array}{rr|rrrrr|rr} k & l & a & b & c & d & e & f & g \\ & & & mk & nk & ok & pk & qk & \\ & & & & ml & nl & ol & pl & ql \\ \hline & & m & n & o & p & q & r & s \end{array}$$

This display clearly suggests an algorithm: Drop down a as m. Multiply m with k and l and write the answers in the next two columns diagonally. Add the second column to get n. Multiply n with k and l and write the answers in the next two columns diagonally. Add the third column to get o, and so on.

However, we quickly realize that when the degree of the divisor exceeds half of the degree of the dividend, this diagonal form is not space-efficient, as shown below when the divisor is changed to $x^5 - hx^4 - ix^3 - jx^2 - kx - l$.

Diagonal Form

$$\begin{array}{ccccc|ccccccc}
h & i & j & k & l & a & b & c & d & e & f & g \\
 & & & & & & mh & nh & & & & \\
 & & & & & & & mi & ni & & & \\
 & & & & & & & & mj & nj & & \\
 & & & & & & & & & mk & nk & \\
 & & & & & & & & & & ml & nl \\
\hline
 & & & & & m & \multicolumn{1}{c|}{n} & o & p & q & r & s
\end{array}$$

We can make this notation more compact by writing all the successive products horizontally.

Horizontal Form

$$\begin{array}{ccccc|ccccccc}
h & i & j & k & l & a & b & c & d & e & f & g \\
 & & & & & & mh & mi & mj & mk & ml & \\
 & & & & & & & nh & ni & nj & nk & nl \\
\hline
 & & & & & m & \multicolumn{1}{c|}{n} & o & p & q & r & s
\end{array}$$

Comparing the two forms, we notice that it really doesn't matter which row each product is written in, as long as it is in the correct column. Therefore, a space-efficient form should employ a 'compact' strategy: *In each column, write the product in the earliest available row.* With this strategy, we can carry out both divisions above in two rows without having to switch forms.

Compact Form

$$\begin{array}{cc|ccccccc}
k & l & a & b & c & d & e & f & g \\
 & & & mk & ml & nl & ol & pl & ql \\
 & & & & nk & ok & pk & qk & \\
\hline
 & & m & n & o & p & \multicolumn{1}{c|}{q} & r & s
\end{array}$$

Compact Form

$$\begin{array}{ccccc|ccccccc}
h & i & j & k & l & a & b & c & d & e & f & g \\
 & & & & & & mh & mi & mj & mk & ml & nl \\
 & & & & & & & nh & ni & nj & nk & \\
\hline
 & & & & & m & \multicolumn{1}{c|}{n} & o & p & q & r & s
\end{array}$$

Still, we would need to know in advance how many rows are needed between the starting row and the answer row. We avoid this issue by writing the products above the starting line, converting the compact strategy into a natural 'piling by gravity.' In an attempt to undermine the dominance of long division of polynomials in school curricula and textbooks, we shall call this compact extension of synthetic division *short division*. For example, the short division of $ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$ by $x^3 - jx^2 - kx - l$ is performed as follows.

Short Division

$$
\begin{array}{ccc|cccc|ccc}
 & & & & & & oj & pj & & \\
 & & & & & nj & nk & ok & pk & \\
 & & & & mj & mk & ml & nl & ol & pl \\
j & k & l & a & b & c & d & e & f & g \\
\hline
 & & & m & n & o & p & q & r & s
\end{array}
$$

For a concrete example, let's divide $3x^6 - 4x^5 + 8x^3 + 9x^2 + 7x - 6$ by $x^3 - 2x^2 + 5$.

$$
\begin{array}{ccc|cccc|ccc}
 & & & & & & 8 & 2 & & \\
 & & & & & 4 & 0 & 0 & 0 & \\
 & & & & 6 & 0 & -15 & -10 & -20 & -5 \\
2 & 0 & -5 & 3 & -4 & 0 & 8 & 9 & 7 & -6 \\
\hline
 & & & 3 & 2 & 4 & 1 & 1 & -13 & -11
\end{array}
$$

Now try to do a long division and you'll find that the quotient is indeed $3x^3 + 2x^2 + 4x + 1$ with remainder $x^2 - 13x - 11$. Notice also that the 0 products could have been omitted to collapse the stacks even further. Try it, it's kind of fun!

Of course, it is very easy to handle the case where the divisor is not monic. For example, $P(x) \div (mx^2 - kx - l)$ has quotient $\frac{1}{m}Q(x)$ and remainder $R(x)$ if $P(x) \div (x^2 - \frac{k}{m}x - \frac{l}{m})$ has quotient $Q(x)$ and remainder $R(x)$.

> Mr. Scogan paid no attention to his denial, but went on: " . . . As for the artist, he is preoccupied with problems that are so utterly unlike those of the ordinary adult man—problems of pure aesthetics which don't so much as present themselves to people like myself—that a description of his mental processes is as boring to the ordinary reader as a piece of pure mathematics . . . ."
>
> —from *Chrome Yellow* by Aldous Huxley