

# Mathematics 2

Mathematics Department  
Phillips Exeter Academy  
Exeter, NH  
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## To the Student

**Contents:** Members of the PEA Mathematics Department have written the material in this book. As you work through it, you will discover that algebra, geometry, and trigonometry have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on tangents to circles. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important theorems. There is no index as such, but the reference section that starts on page 201 should help you recall the meanings of key words that are defined in the problems (where they usually appear italicized).

**Comments on problem-solving:** You should approach each problem as an exploration. Reading each question carefully is essential, especially since definitions, highlighted in italics, are routinely inserted into the problem texts. It is important to make accurate diagrams whenever appropriate. Useful strategies to keep in mind are: create an easier problem, guess and check, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day.

Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

**About technology:** Many of the problems in this book require the use of technology (graphing calculators or computer software) in order to solve them. Moreover, you are encouraged to use technology to explore, and to formulate and test conjectures. Keep the following guidelines in mind: write before you calculate, so that you will have a clear record of what you have done; store intermediate answers in your calculator for later use in your solution; pay attention to the degree of accuracy requested; refer to your calculator's manual when needed; and be prepared to explain your method to your classmates. Also, if you are asked to “graph  $y = (2x - 3)/(x + 1)$ ”, for instance, the expectation is that, although you might use your calculator to generate a picture of the curve, you should sketch that picture in your notebook or on the board, with correctly scaled axes.



Phillips Exeter Academy

Introductory Math Guide for New Students  
(For students, by students!)



## Introduction

Annually, approximately 300 new students take up studies in the Mathematics Department. Coming from various styles of teaching, as a new student you will quickly come to realize the distinct methods and philosophies of teaching at Exeter. One aspect of Exeter that often catches students unaware is the math curriculum. I encourage all new students to come to the math table with a clear mind. You may not grasp, understand, or even like math at first, but you will have to be prepared for anything that comes before you.

During the fall of 2000, the new students avidly voiced a concern about the math curriculum. Our concern ranged from grading, to math policies, and even to the very different teaching styles utilized in the mathematics department. The guide that you have begun reading was written solely by students, with the intent of preparing you for the task that you have embarked upon. This guide includes tips for survival, testimonials of how we felt when entering the math classroom, and aspects of math that we would have liked to have known, before we felt overwhelmed. Hopefully, this guide will ease your transition into math at Exeter. Remember, “Anything worth doing, is hard to do.” Mr. Higgins ’36.

— Anthony L. Riley ’04

*“I learned a lot more by teaching myself than by being taught by someone else.”*

*“One learns many ways to do different problems. Since each problem is different, you are forced to use all aspects of math.”*

*“It takes longer for new concepts to sink in . . . you understand, but because it didn’t sink in, it’s very hard to expand with that concept.”*

*“It makes me think more. The way the math books are setup (i.e. simple problems progressing to harder ones on a concept) really helps me understand the mathematical concepts.”*

*“When you discover or formulate a concept yourself, you remember it better and understand the concept better than if we memorized it or the teacher just told us that the formula was ‘xyz’.”*

## Homework

Math homework = no explanations and eight problems a night. For the most part, it has become standard among most math teachers to give about eight problems a night; but I have even had a teacher who gave ten — though two problems may not seem like a big deal, it can be. Since all the problems are scenarios, and often have topics that vary, they also range in complexity, from a simple, one-sentence question, to a full-fledged paragraph with an eight-part answer! Don’t fret though, transition to homework will come with time, similar to how you gain wisdom, as you get older. Homework can vary greatly from night to night, so be flexible with your time — this leads to another part of doing your homework. IN ALL CLASSES THAT MEET FIVE TIMES A WEEK, INCLUDING MATHEMATICS, YOU SHOULD SPEND 50 MINUTES AT THE MAXIMUM, DOING HOMEWORK! No teacher should ever expect you to spend more time, with the large workload Exonians carry. Try your hardest to concentrate, and utilize those 50 minutes as much as possible.

Without any explanations showing you exactly how to do your homework, how are you supposed to do a problem that you have absolutely no clue about? (This WILL happen!) Ask somebody in your dorm. Another person in your dorm might be in the same class, or the same level, and it is always helpful to seek the assistance of someone in a higher level of math. Also remember, there is a difference between homework and studying; after you're through with the eight problems assigned to you, go back over your work from the last few days.

*“...with homework, you wouldn't get marked down if you didn't do a problem.”*

### **Going to the Board**

It is very important to go to the board to put up homework problems. Usually, every homework problem is put up on the board at the beginning of class, and then they are discussed in class. If you regularly put problems up on the board, your teacher will have a good feel of where you stand in the class; a confident student will most likely be more active in participating in the class.

### **Plagiarism**

One thing to keep in mind is plagiarism. You can get help from almost anywhere, but make sure that you cite your help, and that all work shown or turned in is your own, even if someone else showed you how to do it. Teachers do occasionally give problems/quizzes/tests to be completed at home. You may not receive help on these assessments, unless instructed to by your teacher; it is imperative that all the work is yours.

### **Math Extra-Help**

Getting help is an integral part of staying on top of the math program here at Exeter. It can be rather frustrating to be lost and feel you have nowhere to turn. There are a few tricks of the trade however, which ensure your “safety,” with this possibly overwhelming word problem extravaganza.

### **Teachers and Meetings**

The very first place to turn for help should be your teacher. Since teachers at Exeter have many fewer students than teachers at other schools, they are never less than eager to help you succeed in any way they can. There is actually one designated time slot a week for students to meet with teachers, which is meetings period on Saturday. You can always call or ask a teacher for help. If there is no time during the day, it is always possible to check out of the dorm after your check-in time, to meet with your teacher at their apartment, or house. It is easiest to do this on the nights that your teacher is on duty in his/her dorm. Getting help from your teacher is the first and most reliable source to turn to, for extra help.

*“You could meet with the teacher for extra help anytime.”*

*“Extra help sessions one-on-one with the teacher. My old math text.”*



## 7-9 Math Help

Along with help from your teacher, there are several other places to get help. From 7-9 PM every night, except Saturday, there is a Math and Science help group in the Science Center. Each evening, the lab is filled with students in a broad range of math levels, which should be able to help you with problems you have. Also, remember that your homework is not graded everyday, and your teacher will usually tell you when he/she will be grading a particular assignment. This means that you can always find someone in your dorm that will help you catch up or simply help you with a tough problem. If you are a day student, I would definitely recommend going to Science and Math Help.

*“...harder to understand concepts if you don’t understand a problem because each problem is trying to teach you something different that leads to a new concept.”*

*“Hard to separate different math concepts. Not sure what kind of math it is I’m learning. More difficult to review.”*

## Different Teachers Teach Differently

The teachers at Exeter usually develop their own style of teaching, fitted to their philosophy of the subject they teach; it is no different in the math department. Teachers vary at all levels: they grade differently, assess your knowledge differently, teach differently, and go over homework differently. They offer help differently, too. This simply means that it is essential that you be prepared each term to adapt to a particular teaching style. For instance, my teacher tests me about every two weeks, gives hand-in problems every couple of days, and also gives a few quizzes. However, my friend, who is in the same level math as I am, has a teacher who doesn’t give any tests or quizzes; he only grades on class participation, and assigns a single hand-in problem, each assignment. Don’t be afraid to ask your teacher how they grade, because this can become very crucial; various teachers put more weight on class participation in grading while others do the opposite. You must learn to be flexible to teaching styles and even your teacher’s personality. This is a necessity for all departments at Exeter, including math.

*“The tests are the hardest part between terms to adapt to, but if you prepare well, there shouldn’t be a problem.”*

*“Tests are hard. Can’t go at your own pace.”*

*“My other teacher taught and pointed out which problems are related when they are six pages apart.”*

*“It took a few days adjusting to, but if you pay attention to what the teacher says and ask him/her questions about their expectations, transitions should be smooth.”*

*“Inconsistent. Every teacher gave different amounts of homework and tests. Class work varied too. My fall term teacher made us put every problem on the board, whereas my winter term teacher only concentrated on a few.”*

— Jonathan Barbee ’04

— Ryan Levihn-Coon ’04

## New Student Testimonials

*“There was not a foundation to build on. There were no ‘example’ problems.”*

After eight years of math textbooks and lecture-style math classes, math at Exeter was a lot to get used to. My entire elementary math education was based on reading how to do problems from the textbook, then practicing monotonous problems that had no real-life relevance, one after the other. This method is fine for some people, but it wasn’t for me. By the time I came to Exeter, I was ready for a change of pace, and I certainly got one.

Having somewhat of a background in algebra, I thought the Transition 1 course was just right for me. It went over basic algebra and problem-solving techniques. The math books at Exeter are very different from traditional books. They are compiled by the teachers, and consist of pages upon pages of word problems that lead you to find your own methods of solving problems. The problems are not very instructional, they lay the information down for you, most times introducing new vocabulary, (there is an index in the back of the book), and allow you to think about the problem, and solve it any way that you can. When I first used this booklet, I was a little thrown back; it was so different from everything I had done before — but by the time the term was over, I had the new method down.

The actual math classes at Exeter were hard to get used to as well. Teachers usually assign about eight problems a night, leaving you time to “explore” the problems and give each one some thought. Then, next class, students put all the homework problems on the board. The class goes over each problem; everyone shares their method and even difficulties that they ran into while solving it. I think the hardest thing to get used to, is being able to openly ask questions. No one wants to be wrong, I guess it is human nature, but in the world of Exeter math, you can’t be afraid to ask questions. You have to seize the opportunity to speak up and say “I don’t understand,” or “How did you get that answer?” If you don’t ask questions, you will never get the answers you need to thrive.

Something that my current math teacher always says is to make all your mistakes on the board, because when a test comes around, you don’t want to make mistakes on paper. This is so true, class time is practice time, and it’s hard to get used to not feeling embarrassed after you answer problems incorrectly. You need to go out on a limb and try your best. If you get a problem wrong on the board, it’s one new thing learned in class, not to mention, one less thing to worry about messing up on, on the next test.

Math at Exeter is really based on cooperation, you, your classmates, and your teacher. It takes a while to get used to, but in the end, it is worth the effort.

— Hazel Cipolle ’04

*“At first, I was very shy and had a hard time asking questions.*

*“Sometimes other students didn’t explain problems clearly.”*

*“Solutions to certain problems by other students are sometimes not the fastest or easiest.  
Some students might know tricks and special techniques that aren’t covered.”*

I entered my second math class of Fall Term as a ninth grader, with a feeling of dread. Though I had understood the homework the night before, I looked down at my paper with a blank mind, unsure how I had done any of the problems. The class sat nervously around the table until we were prompted by the teacher to put the homework on the board. One boy stood up and picked up some chalk. Soon others followed suit. I stayed glued to my seat with the same question running through my mind, what if I get it wrong?

I was convinced that everyone would make fun of me, that they would tear my work apart, that each person around that table was smarter than I was. I soon found that I was the only one still seated and hurried to the board. The only available problem was one I was slightly unsure of. I wrote my work quickly and reclaimed my seat.

We reviewed the different problems, and everyone was successful. I explained my work and awaited the class’ response. My classmates agreed with the bulk of my work, though there was a question on one part. They suggested different ways to find the answer and we were able to work through the problem, together.

I returned to my seat feeling much more confident. Not only were my questions cleared up, but my classmates’ questions were answered as well. Everyone benefited.

I learned one of the more important lessons about math at Exeter that day; it doesn’t matter if you are right or wrong. Your classmates will be supportive of you, and tolerant of your questions. Chances are, if you had trouble with a problem, someone else in the class did too. Another thing to keep in mind is that the teacher expects nothing more than that you try to do a problem to the best of your ability. If you explain a problem that turns out to be incorrect, the teacher will not judge you harshly. They understand that no one is always correct, and will not be angry or upset with you.

— Elisabeth Ramsey ’04

*“My background in math was a little weaker than most people’s, therefore I was unsure how to do many of the problems. I never thoroughly understood how to do a problem before I saw it in the book.”*

I never thought math would be a problem. That is, until I came to Exeter. I entered into Math T1B, clueless as to what the curriculum would be. The day I bought the Math One book from the Bookstore Annex, I stared at the problems in disbelief. ALL WORD PROBLEMS. “Why word problems?” I thought. I had dreaded word problems ever since I was a second grader, and on my comments it always read, “Charly is a good math student, but she needs to work on word problems.” I was in shock. I would have to learn math in an entirely new language. I began to dread my B format math class.

My first math test at Exeter was horrible. I had never seen a D– on a math test. Never. I was upset and I felt dumb, especially since others in my class got better grades, and because my roommate was extremely good in math. I cried. I said I wanted to go home where things were easier. But finally I realized, “I was being given a challenge. I had to at least try.”

I went to my math teacher for extra help. I asked questions more often (though not as much as I should have), and slowly I began to understand the problems better. My grades gradually got better, by going from a D– to a C+ to a B and eventually I got an A–. It was hard, but that is Exeter. You just have to get passed that first hump, though little ones will follow. As long as you don’t compare yourself to others, and you ask for help when you need it, you should get used to the math curriculum. I still struggle, but as long as I don’t get intimidated and don’t give up, I am able to bring my grades up.

— Charly Simpson ’04

*The above quotes in italics were taken from a survey of new students in the spring of 2001.*

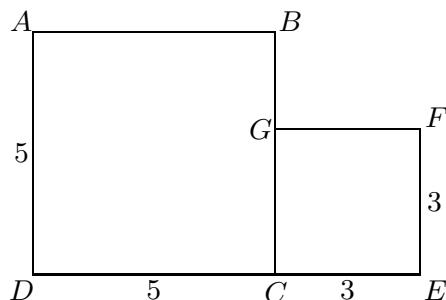
## Mathematics 2

1. A  $5 \times 5$  square and a  $3 \times 3$  square can be cut into pieces that will fit together to form a third square.

(a) Find the length of a side of the third square.

(b) In the diagram at right, mark  $P$  on segment  $DC$  so that  $PD = 3$ , then draw segments  $PA$  and  $PF$ . Calculate the lengths of these segments.

(c) Segments  $PA$  and  $PF$  divide the squares into pieces. Arrange the pieces to form the third square.



2. (Continuation) Change the sizes of the squares to  $AD = 8$  and  $EF = 4$ , and redraw the diagram. Where should point  $P$  be marked this time? Form the third square again.

3. (Continuation) Will the preceding method *always* produce pieces that form a new square? If your answer is *yes*, prepare a written explanation. If your answer is *no*, provide a counterexample — two specific squares that can *not* be converted to a single square.

4. Instead of walking along two sides of a rectangular field, Fran took a shortcut along the diagonal, thus saving distance equal to half the length of the longer side. Find the length of the long side of the field, given that the length of the short side is 156 meters.

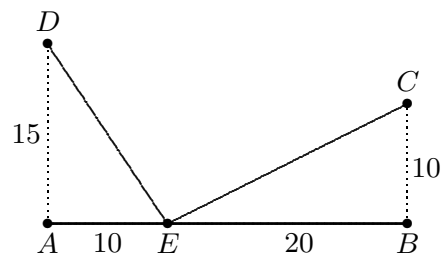
5. Let  $A = (0, 0)$ ,  $B = (7, 1)$ ,  $C = (12, 6)$ , and  $D = (5, 5)$ . Plot these points and connect the dots to form the quadrilateral  $ABCD$ . Verify that all four sides have the same length. Such a figure is called *equilateral*.

6. The main use of the Pythagorean Theorem is to find distances. Originally (6<sup>th</sup> century BC), however, it was regarded as a statement about *areas*. Explain this interpretation.

7. Two iron rails, each 50 feet long, are laid end to end with no space between them. During the summer, the heat causes each rail to increase in length by 0.04 percent. Although this is a small increase, the lack of space at the joint makes the joint buckle upward. What distance upward will the joint be forced to rise? [Assume that each rail *remains straight*, and that the other ends of the rails are anchored.]

8. In the diagram,  $AEB$  is straight and angles  $A$  and  $B$  are right. Calculate the total distance  $DE + EC$ .

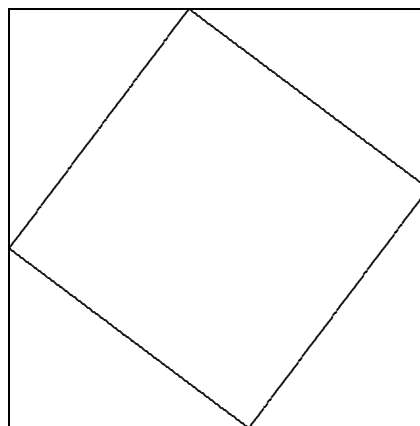
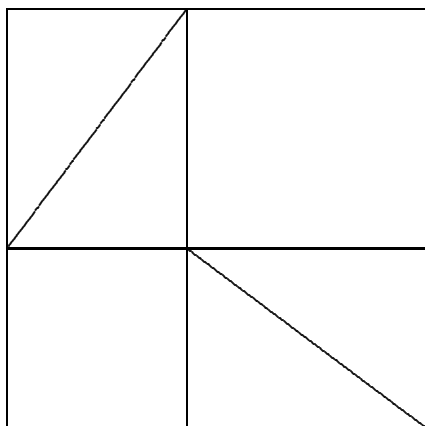
9. (Continuation) If  $AE = 20$  and  $EB = 10$  instead, would  $DE + EC$  be the same?



10. (Continuation) You have seen that the value chosen for  $AE$  *determines* the value of  $DE + EC$ . One also says that  $DE + EC$  *is a function of*  $AE$ . Letting  $x$  stand for  $AE$  (and  $30 - x$  for  $EB$ ), write a formula for this function. Then enter this formula into your calculator, graph it, and find the value of  $x$  that produces the *shortest* path from  $D$  to  $C$  through  $E$ . Draw an accurate picture of *this* path, and make a conjecture about angles  $AED$  and  $BEC$ . Use your protractor to test your conjecture.

## Mathematics 2

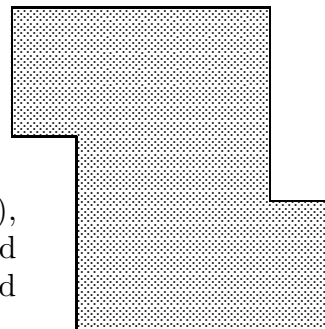
1. Two different points on the line  $y = 2$  are each exactly 13 units from the point  $(7, 14)$ . Draw a picture of this situation, and then find the coordinates of these points.
2. Give an example of a point that is the same distance from  $(3, 0)$  as it is from  $(7, 0)$ . Find lots of examples. Describe the configuration of all such points. In particular, how does this configuration relate to the two given points?
3. Verify that the hexagon formed by  $A = (0, 0)$ ,  $B = (2, 1)$ ,  $C = (3, 3)$ ,  $D = (2, 5)$ ,  $E = (0, 4)$ , and  $F = (-1, 2)$  is equilateral. Is it also *equiangular*?
4. Draw a 20-by-20 square  $ABCD$ . Mark  $P$  on  $AB$  so that  $AP = 8$ ,  $Q$  on  $BC$  so that  $BQ = 5$ ,  $R$  on  $CD$  so that  $CR = 8$ , and  $S$  on  $DA$  so that  $DS = 5$ . Find the lengths of the sides of quadrilateral  $PQRS$ . Is there anything special about this quadrilateral? Explain.
5. Verify that  $P = (1, -1)$  is the same distance from  $A = (5, 1)$  as it is from  $B = (-1, 3)$ . It is customary to say that  $P$  is *equidistant* from  $A$  and  $B$ . Find three more points that are equidistant from  $A$  and  $B$ . By the way, to “find” a point means to find its *coordinates*. Can points equidistant from  $A$  and  $B$  be found in every *quadrant*?
6. The two-part diagram below, which shows two different dissections of the same square, was designed to help *prove* the Pythagorean Theorem. Provide the missing details.



7. Inside a 5-by-5 square, it is possible to place four 3-4-5 triangles so that they do not overlap. Show how. Then explain why you can be sure that it is impossible to squeeze in a fifth triangle of the same size.
8. If you were writing a geometry book, and you had to define a mathematical figure called a *kite*, how would you word your definition?
9. Find both points on the line  $y = 3$  that are 10 units from  $(2, -3)$ .
10. On a number line, where is  $\frac{1}{2}(p + q)$  in relation to  $p$  and  $q$ ?

## Mathematics 2

1. Some terminology: Figures that have exactly the same shape and size are called *congruent*. Dissect the region shown at right into two congruent parts. How many different ways of doing this can you find?



2. Let  $A = (2, 4)$ ,  $B = (4, 5)$ ,  $C = (6, 1)$ ,  $T = (7, 3)$ ,  $U = (9, 4)$ , and  $V = (11, 0)$ . Triangles  $ABC$  and  $TUV$  are specially related to each other. Make calculations to clarify this statement, and write a few words to describe what you discover.

3. A triangle that has two sides of equal length is called *isosceles*. Make up an example of an isosceles triangle, one of whose vertices is  $(3, 5)$ . If you can, find a triangle that does not have any horizontal or vertical sides.

4. Una recently purchased two boxes of ten-inch candles — one box from a discount store, and the other from an expensive boutique. It so happens that the inexpensive candles last only three hours each, while the expensive candles last five hours each. One evening, Una hosted a dinner party and lighted two candles — one from each box — at 7:30 pm. During dessert, a guest noticed that one candle was twice as long as the other. At what time was this observation made?

5. Let  $A = (1, 5)$  and  $B = (3, -1)$ . Verify that  $P = (8, 4)$  is equidistant from  $A$  and  $B$ . Find at least two more points that are equidistant from  $A$  and  $B$ . Describe all such points.

6. Find two points on the  $y$ -axis that are 9 units from  $(7, 5)$ .

7. A *lattice point* is a point whose coordinates are *integers*. Find two lattice points that are exactly  $\sqrt{13}$  units apart. Is it possible to find lattice points that are  $\sqrt{15}$  units apart? Is it possible to form a square whose area is 18 by connecting four lattice points? Explain.

*Some terminology:* When two angles fit together to form a straight angle (a 180-degree angle, in other words), they are called *supplementary angles*, and either angle is the *supplement* of the other. When an angle is the same size as its supplement (a 90-degree angle), it is called a *right angle*. When two angles fit together to form a right angle, they are called *complementary angles*, and either angle is the *complement* of the other. Two lines that form a right angle are said to be *perpendicular*.

8. The three angles of a triangle fit together to form a straight angle. In one form or another, this statement is a fundamental *postulate* of Euclidean geometry — accepted as true, without proof. Taking this for granted, then, what can be said about the two non-right angles in a right triangle?

9. Let  $P = (a, b)$ ,  $Q = (0, 0)$ , and  $R = (-b, a)$ , where  $a$  and  $b$  are positive numbers. Prove that angle  $PQR$  is right, by introducing two congruent right triangles into your diagram. Verify that the slope of segment  $QP$  is the *negative reciprocal* of the slope of segment  $QR$ .

## Mathematics 2

1. Find an example of an equilateral hexagon whose sides are all  $\sqrt{13}$  units long. Give coordinates for all six points.

2. I have been observing the motion of a bug that is crawling on my graph paper. When I started watching, it was at the point  $(1, 2)$ . Ten seconds later it was at  $(3, 5)$ . Another ten seconds later it was at  $(5, 8)$ . After another ten seconds it was at  $(7, 11)$ .

(a) Draw a picture that illustrates what is happening.

(b) Write a description of any pattern that you notice. What assumptions are you making?

(c) Where was the bug 25 seconds after I started watching it?

(d) Where was the bug 26 seconds after I started watching it?

3. The point on segment  $AB$  that is equidistant from  $A$  and  $B$  is called the *midpoint* of  $AB$ . For each of the following, find coordinates for the midpoint of  $AB$ :

(a)  $A = (-1, 5)$  and  $B = (5, -7)$

(b)  $A = (m, n)$  and  $B = (k, l)$

4. Write a formula for the distance from  $A = (-1, 5)$  to  $P = (x, y)$ , and another formula for the distance from  $P = (x, y)$  to  $B = (5, 2)$ . Then write an equation that says that  $P$  is equidistant from  $A$  and  $B$ . Simplify your equation to linear form. This line is called the *perpendicular bisector* of  $AB$ . Verify this by calculating two slopes and one midpoint.

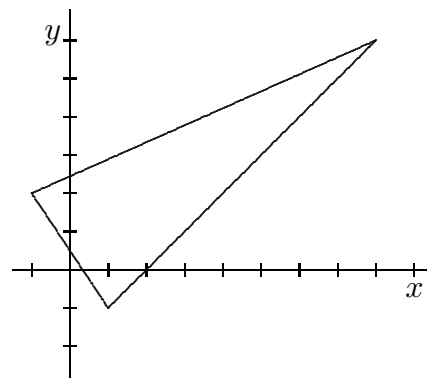
5. Find the slope of the line through

(a)  $(3, 1)$  and  $(3 + 4t, 1 + 3t)$

(b)  $(m - 5, n)$  and  $(5 + m, n^2)$

6. Is it possible for a line  $ax + by = c$  to lack a  $y$ -intercept? To lack an  $x$ -intercept? Explain.

7. The sides of the triangle at right are formed by the graphs of  $3x + 2y = 1$ ,  $y = x - 2$ , and  $-4x + 9y = 22$ . Is the triangle isosceles? How do you know?



8. Pat races at 10 miles per hour, while Kim races at 9 miles per hour. When they both ran in the same long-distance race last week, Pat finished 8 minutes ahead of Kim. What was the length of the race, in miles? Briefly describe your reasoning.

9. (Continuation) Assume that Pat and Kim run at  $p$  and  $k$  miles per hour, respectively, and that Pat finishes  $m$  minutes before Kim. Find the length of the race, in miles.

10. A bug moves linearly with constant speed across my graph paper. I first notice the bug when it is at  $(3, 4)$ . It reaches  $(9, 8)$  after two seconds and  $(15, 12)$  after four seconds.

(a) Predict the position of the bug after six seconds; after nine seconds; after  $t$  seconds.

(b) Is there a time when the bug is equidistant from the  $x$ - and  $y$ -axes? If so, where is it?

11. What is the relation between the lines described by the equations  $-20x + 12y = 36$  and  $-35x + 21y = 63$ ? Find a third equation in the form  $ax + by = 90$  that fits this pattern.



## Mathematics 2

1. Rewrite the equation  $3x - 5y = 30$  in the form  $ax + by = 1$ . Are there lines whose equations *cannot* be rewritten in this form?

2. Consider the linear equation  $y = 3.62(x - 1.35) + 2.74$ .

(a) What is the slope of this line?

(b) What is the value of  $y$  when  $x = 1.35$ ?

(c) This equation is written in *point-slope* form. Explain the terminology.

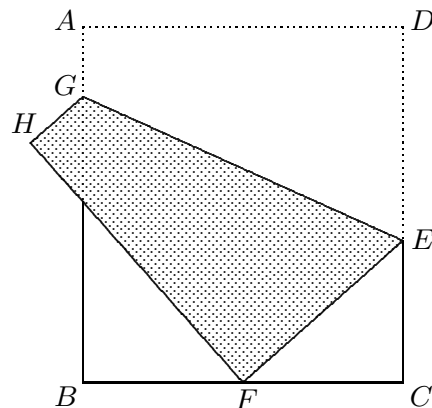
(d) Use your calculator to graph this line.

(e) Find an equation for the line through  $(4.23, -2.58)$  that is parallel to this line.

(f) Describe how to use your calculator to graph a line that has slope  $-1.25$  and that goes through the point  $(-3.75, 8.64)$ .

3. The dimensions of rectangular piece of paper  $ABCD$  are  $AB = 10$  and  $BC = 9$ . It is folded so that corner  $D$  is matched with a point  $F$  on edge  $BC$ . Given that length  $DE = 6$ , find  $EF$ ,  $EC$ , and  $FC$ .

4. (Continuation) The lengths  $EF$ ,  $EC$ , and  $FC$  are all functions of the length  $DE$ . The area of triangle  $EFC$  is also a function of  $DE$ . Using  $x$  to stand for  $DE$ , write formulas for these four functions.



5. (Continuation) Find the value of  $x$  that maximizes the area of triangle  $EFC$ .

6. The  $x$ - and  $y$ -coordinates of a point are given by the equations shown below. The position of the point depends on the value assigned to  $t$ . Use your graph paper to plot points corresponding to the values  $t = -4, -3, -2, -1, 0, 1, 2, 3$ , and  $4$ . Do you recognize any patterns? Describe them.

$$\begin{cases} x = 2 + 2t \\ y = 5 - t \end{cases}$$

7. Plot the following points on the coordinate plane:  $(1, 2)$ ,  $(2, 5)$ ,  $(3, 8)$ . Write equations, similar to those in the preceding exercise, that produce these points when  $t$ -values are assigned. There is more than one correct answer.

8. Given that  $2x - 3y = 17$  and  $4x + 3y = 7$ , and without using paper, pencil, or calculator, find the value of  $x$ .

9. A slope can be considered to be a *rate*. Explain this interpretation.

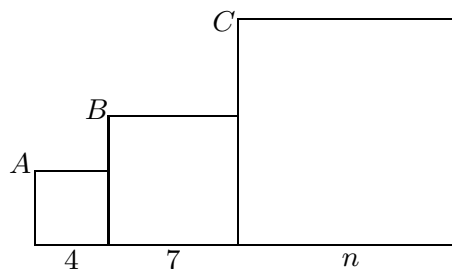
10. Find  $a$  and  $b$  so that  $ax + by = 1$  has  $x$ -intercept 5 and  $y$ -intercept 8.

11. Given points  $A = (-2, 7)$  and  $B = (3, 3)$ , find two points  $P$  that are on the perpendicular bisector of  $AB$ . In each case, what can you say about segments  $PA$  and  $PB$ ?

12. Explain the difference between a line that has *no slope* and a line whose slope is *zero*.

## Mathematics 2

1. Three squares are placed next to each other as shown. The vertices  $A$ ,  $B$ , and  $C$  are collinear. Find the dimension  $n$ .



2. (Continuation) Replace the lengths 4 and 7 by  $m$  and  $k$ , respectively. Express  $k$  in terms of  $m$  and  $n$ .

3. A five-foot Prep casts a shadow that is 40 feet long while standing 200 feet from a streetlight. How high above the ground is the lamp?

4. (Continuation) How far from the streetlight should the Prep stand, in order to cast a shadow that is exactly as long as the Prep is tall?

5. An airplane 27000 feet above the ground begins descending at the rate of 1500 feet per minute. Assuming the plane continues at the same rate of descent, how long will it be before it is on the ground?

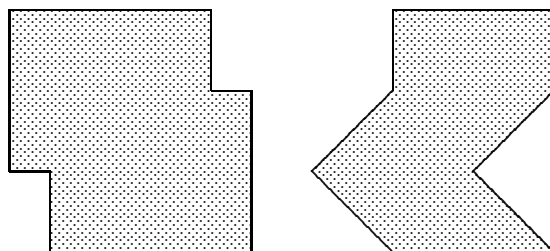
6. (Continuation) Graph the line  $y = 27000 - 1500x$ , using an appropriate window on your calculator. With the preceding problem in mind, explain the significance of the slope of this line and its two intercepts.

7. An airplane is flying at 36000 feet directly above Lincoln, Nebraska. A little later the plane is flying at 28000 feet directly above Des Moines, Iowa, which is 160 miles from Lincoln. Assuming a constant rate of descent, predict how far from Des Moines the airplane will be when it lands.

8. In a dream, Blair is confined to a coordinate plane, moving along a line with a constant speed. Blair's position at 4 am is  $(2, 5)$  and at 6 am it is  $(6, 3)$ . What is Blair's position at 8:15 am when the alarm goes off?

9. Find a way to show that points  $A = (-4, -1)$ ,  $B = (4, 3)$ , and  $C = (8, 5)$  are collinear.

10. Find as many ways as you can to dissect each figure at right into two congruent parts.



11. Let  $A = (4, 2)$ ,  $B = (11, 6)$ ,  $C = (7, 13)$ , and  $D = (0, 9)$ . Show that  $ABCD$  is a square.

12. Lynn takes a step, measures its length and obtains 3 feet. Lynn uses this measurement in attempting to pace off a 1-mile course, but the result is 98 feet too long. What is the actual length of Lynn's stride, and how could Lynn have done a more accurate job?

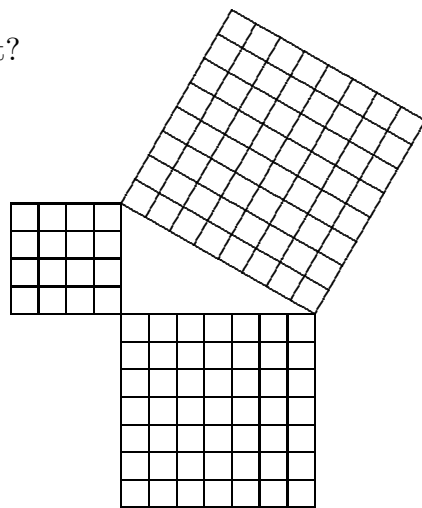
## Mathematics 2

1. One of the legs of a right triangle is 12 units long. The other leg is  $b$  units long and the hypotenuse  $c$  units long, where  $b$  and  $c$  are both integers. Find  $b$  and  $c$ . *Hint:* Both sides of the equation  $c^2 - b^2 = 144$  can be factored.

2. Is there anything wrong with the figure shown at right?

3. Show that a 9-by-16 rectangle can be transformed into a square by dissection. In other words, the rectangle can be cut into pieces that can be reassembled to form the square. Do it with as few pieces as possible.

4. At noon one day, Corey decided to follow a straight course in a motor boat. After one hour of making no turns and traveling at a steady rate, the boat was 6 miles east and 8 miles north of its point of departure. What was Corey's position at two o'clock? How far had Corey traveled? What was Corey's speed?



5. (Continuation) Assume that the fuel tank initially held 12 gallons, and that the boat gets 4 miles to the gallon. How far did Corey get before running out of fuel? When did this happen? When radioing the Coast Guard for help, how should Corey describe the boat's position?

6. Suppose that numbers  $a$ ,  $b$ , and  $c$  fit the equation  $a^2 + b^2 = c^2$ , with  $a = b$ . Express  $c$  in terms of  $a$ . Draw a good picture of such a triangle. What can be said about its angles?

7. The Krakow airport is 3 km west and 5 km north of the city center. At 1 pm, Zuza took off in a Cessna 730. Every six minutes, the plane's position changed by 9 km east and 7 km north. At 2:30 pm, Zuza was flying over the town of Jozefow. In relation to the center of Krakow, (a) where is Jozefow? (b) where was Zuza after  $t$  hours of flying?

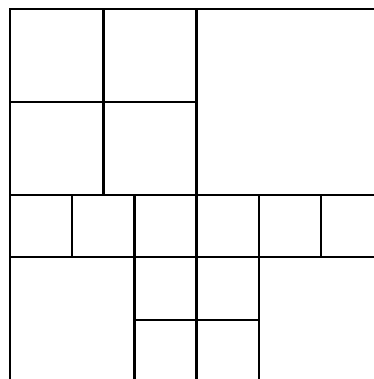
8. Golf balls cost \$0.90 each at Jerzy's Club, which has an annual \$25 membership fee. At Rick & Tom's sporting-goods store, the price is \$1.35 per ball for the same brand. Where you buy your golf balls depends on how many you wish to buy. Explain, and illustrate your reasoning by drawing a graph.

9. Draw the following segments. What do they have in common?  
from  $(3, -1)$  to  $(10, 3)$ ;      from  $(1.3, 0.8)$  to  $(8.3, 4.8)$ ;      from  $(\pi, \sqrt{2})$  to  $(7 + \pi, 4 + \sqrt{2})$ .

10. (Continuation) The *directed segments* have the *same* length and the *same* direction. Each represents the *vector*  $[7, 4]$ . The *components* of the vector are the numbers 7 and 4.  
(a) Find another example of a directed segment that represents this vector. The initial point of your segment is called the *tail* of the vector, and the final point is called the *head*.  
(b) Which of the following directed segments represents  $[7, 4]$ ? from  $(-2, -3)$  to  $(5, -1)$ ; from  $(-3, -2)$  to  $(11, 6)$ ; from  $(10, 5)$  to  $(3, 1)$ ; from  $(-7, -4)$  to  $(0, 0)$ .

## Mathematics 2

1. Is it possible for a positive number to exceed its reciprocal by exactly 1? One number that comes close is  $\frac{8}{5}$ , because  $\frac{8}{5} - \frac{5}{8}$  is  $\frac{39}{40}$ . Is there a fraction that comes closer?
2. Points  $(x, y)$  described by the equations  $x = 1 + 2t$  and  $y = 3 + t$  form a line. Is the point  $(7, 6)$  on this line? How about  $(-3, 1)$ ? How about  $(6, 5.5)$ ? How about  $(11, 7)$ ?
3. The perimeter of an isosceles right triangle is 24 cm. How long are its sides?
4. The  $x$ - and  $y$ -coordinates of a point are given by the equations shown below. Use your graph paper to plot points corresponding to  $t = -1, 0$ , and  $2$ . These points should appear to be collinear. Convince yourself that this is the case, and calculate the slope of this line. The displayed equations are called *parametric*, and  $t$  is called a *parameter*. How is the slope of a line determined from its parametric equations?
 
$$\begin{cases} x = -4 + 3t \\ y = 1 + 2t \end{cases}$$
5. Find parametric equations to describe the line that goes through the points  $A = (5, -3)$  and  $B = (7, 1)$ . There is more than one correct answer to this question.
6. Show that the triangle formed by the lines  $y = 2x - 7$ ,  $x + 2y = 16$ , and  $3x + y = 13$  is isosceles. Show also that the lengths of the sides of this triangle fit the Pythagorean equation. Can you identify the right angle just by looking at the equations?
7. Leaving home on a recent business trip, Kyle drove 10 miles south to reach the airport, then boarded a plane that flew a straight course — 6 miles east and 3 miles north each minute. What was the airspeed of the plane? After two minutes of flight, Kyle was directly above the town of Greenup. How far is Greenup from Kyle's home? A little later, the plane flew over Kyle's birthplace, which is 50 miles from home. When did this occur?
8. A triangle has vertices  $A = (1, 2)$ ,  $B = (3, -5)$ , and  $C = (6, 1)$ . Triangle  $A'B'C'$  is obtained by *sliding* triangle  $ABC$  5 units to the right (in the positive  $x$ -direction, in other words) and 3 units up (in the positive  $y$ -direction). It is also customary to say that vector  $[5, 3]$  has been used to *translate* triangle  $ABC$ . What are the coordinates of  $A'$ ,  $B'$ , and  $C'$ ? By the way, "A prime" is the usual way of reading  $A'$ .
9. (Continuation) When vector  $[h, k]$  is used to translate triangle  $ABC$ , it is found that the image of vertex  $A$  is  $(-3, 7)$ . What are the images of vertices  $B$  and  $C$ ?
10. It is a simple matter to divide a square into four smaller squares, and — as the figure at right shows — it is also possible to divide a square into seventeen smaller squares. In addition to four and seventeen, what numbers of smaller squares are possible? The smaller squares can be of any size whatsoever, as long as they fit neatly together to form one large square.



## Mathematics 2

1. Caught in another nightmare, Blair is moving along the line  $y = 3x + 2$ . At midnight, Blair's position is  $(1, 5)$ , the  $x$ -coordinate increasing by 4 units every hour. Write parametric equations that describe Blair's position  $t$  hours after midnight. What was Blair's position at 10:15 pm when the nightmare started? Find Blair's speed, in units per hour.

2. The parametric equations  $x = -2 - 3t$  and  $y = 6 + 4t$  describe the position of a particle, in meters and seconds. How does the particle's position change each second? each minute? What is the speed of the particle, in meters per second? Write parametric equations that describe the particle's position, using meters and *minutes* as units.

3. Let  $A = (1, 2)$ ,  $B = (5, 1)$ ,  $C = (6, 3)$ , and  $D = (2, 5)$ . Let  $P = (-1, -1)$ ,  $Q = (3, -2)$ ,  $R = (4, 0)$ , and  $S = (0, 2)$ . Use a vector to describe how quadrilateral  $ABCD$  is related to quadrilateral  $PQRS$ .

4. Let  $K = (3, 8)$ ,  $L = (7, 5)$ , and  $M = (4, 1)$ . Find coordinates for the vertices of the triangle that is obtained by using the vector  $[2, -5]$  to slide triangle  $KLM$ . How far does each vertex slide?

5. Find parametric equations that describe the following lines:  
**(a)** through  $(3, 1)$  and  $(7, 3)$       **(b)** through  $(7, -1)$  and  $(7, 3)$

6. Find all points on the  $y$ -axis that are twice as far from  $(-5, 0)$  as they are from  $(1, 0)$ . Begin by making a drawing and estimating. Find all such points on the  $x$ -axis. In each case, how many points did you find? How do you know that you have found them all?

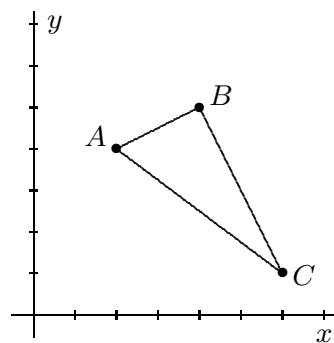
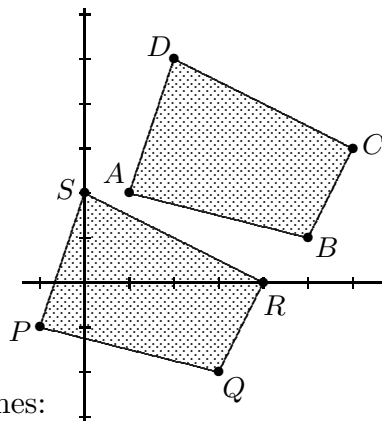
7. Let  $A = (-5, 0)$ ,  $B = (5, 0)$ , and  $C = (2, 6)$ ; let  $K = (5, -2)$ ,  $L = (13, 4)$ , and  $M = (7, 7)$ . Verify that the length of each side of triangle  $ABC$  matches the length of a side of triangle  $KLM$ . Because of this data, it is natural to regard the triangles as being in some sense equivalent. It is customary to call the triangles *congruent*. The basis used for this judgment is called the *side-side-side* criterion. What can you say about the sizes of angles  $ACB$  and  $KML$ ? What is your reasoning? What about the other angles?

8. (Continuation) Are the triangles related by a vector translation? Why or why not?

9. Let  $A = (2, 4)$ ,  $B = (4, 5)$ , and  $C = (6, 1)$ . Triangle  $ABC$  is shown at right. Draw three new triangles as follows:

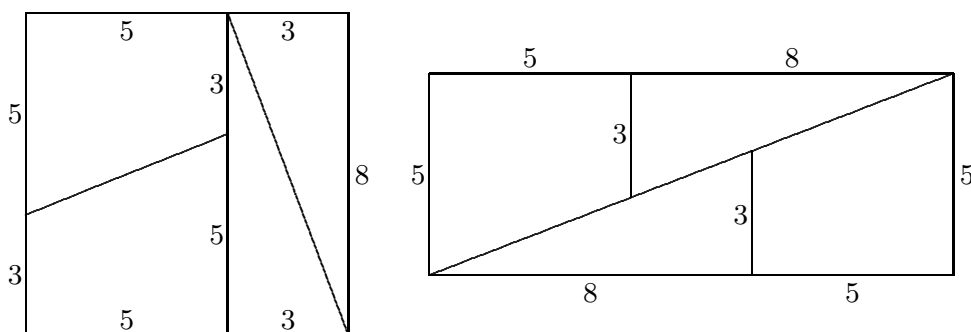
- (a)**  $\triangle PQR$  has  $P = (11, 1)$ ,  $Q = (10, -1)$ , and  $R = (6, 1)$ ;
- (b)**  $\triangle KLM$  has  $K = (8, 10)$ ,  $L = (7, 8)$ , and  $M = (11, 6)$ ;
- (c)**  $\triangle TUV$  has  $T = (-2, 6)$ ,  $U = (0, 5)$ , and  $V = (2, 9)$ .

These triangles are not obtained from  $ABC$  by applying vector translations. Instead, each of the appropriate transformations is described by one of the suggestive names *reflection*, *rotation*, or *glide-reflection*. Decide which is which, with justification.



## Mathematics 2

1. In baseball, the bases are placed at the corners of a square whose sides are 90 feet long. Home plate and second base are at opposite corners. To the nearest eighth of an inch, how far is it from home plate to second base?
2. A bug is moving along the line  $3x + 4y = 12$  with constant speed 5 units per second. The bug crosses the  $x$ -axis when  $t = 0$  seconds. It crosses the  $y$ -axis later. When? Where is the bug when  $t = 2$ ? when  $t = -1$ ? when  $t = 1.5$ ? What does a negative  $t$ -value mean?
3. Give the components of the vector whose length is 10 and whose direction *opposes* the direction of  $[-4, 3]$ .
4. Find parametric equations to describe the line  $3x + 4y = 12$ . Use your equations to find coordinates for the point that is three-fifths of the way from  $(4, 0)$  to  $(0, 3)$ . By calculating some distances, verify that you have the correct point.
5. A 9-by-12 rectangular picture is framed by a border of uniform width. Given that the combined area of picture plus frame is 180 square units, find the width of the border.
6. Let  $A = (0, 0)$ ,  $B = (2, -1)$ ,  $C = (-1, 3)$ ,  $P = (8, 2)$ ,  $Q = (10, 3)$ , and  $R = (5, 3)$ . Plot these points. Angles  $BAC$  and  $QPR$  should look like they are the same size. Find evidence to support this conclusion.
7. An equilateral quadrilateral is called a *rhombus*. A square is a simple example of a rhombus. Find a non-square rhombus whose diagonals and sides are *not* parallel to the rulings on your graph paper. Use coordinates to describe its vertices. Write a brief description of the process you used to find your example.
8. Using a ruler and protractor, draw a triangle that has an 8-cm side and a 6-cm side, which make a 30-degree angle. This is a *side-angle-side* description. Cut out the figure so that you can compare triangles with your classmates. Will your triangles be congruent?
9. Compare the two figures shown below. Is there anything wrong with what you see?



10. Tracy and Kelly are running laps on the indoor track — at steady speeds, but in opposite directions. They meet every 20 seconds. It takes Tracy 45 seconds to complete each lap. How many seconds does it take for each of Kelly's laps? Check your answer.

## Mathematics 2

1. Instead of saying that Remy moves *3 units left and 2 units up*, you can say that Remy's position is *displaced* by the vector  $[-3, 2]$ . Identify the following displacement vectors:
  - (a) Forrest starts at  $(2, 3)$  at 1 pm, and has moved to  $(5, 9)$  by 6 am;
  - (b) at noon, Eugene is at  $(3, 4)$ ; two hours earlier Eugene was at  $(6, 2)$ ;
  - (c) during a single hour, a small airplane flew 40 miles north and 100 miles west.
2. Kirby moves with constant speed 5 units per hour along the line  $y = \frac{3}{4}x + 6$ , crossing the  $y$ -axis at midnight and the  $x$ -axis later. When is the  $x$ -axis crossing made? What does it mean to say that *Kirby's position is a function of time*? What is Kirby's position 1.5 hours after midnight? What is Kirby's position  $t$  hours after midnight?
3. A bug is initially at  $(-3, 7)$ . Where is the bug after being displaced by vector  $[-7, 8]$ ?
4. With the aid of a ruler and protractor, draw a triangle that has an 8-cm side, a 6-cm side, and a 45-degree angle that is *not* formed by the two given sides. This is a *side-side-angle* description. Cut out the figure so that you can compare triangles with your classmates. Do you expect your triangles to be congruent?
5. Plot points  $K = (0, 0)$ ,  $L = (7, -1)$ ,  $M = (9, 3)$ ,  $P = (6, 7)$ ,  $Q = (10, 5)$ , and  $R = (1, 2)$ . Show that the triangles  $KLM$  and  $RPQ$  are congruent. Show also that neither triangle is a vector translation of the other. Describe how one triangle has been transformed into the other.
6. (Continuation) If two figures are congruent, then their parts *correspond*. In other words, each part of one figure has been matched with a definite part of the other figure. In the triangle  $PQR$ , which angle corresponds to angle  $M$ ? Which side corresponds to  $KL$ ? In general, what can be said about corresponding parts of congruent figures? How might you confirm your hunch experimentally?
7. What is the slope of the line  $ax + by = c$ ? Find an equation for the line through the origin that is perpendicular to the line  $ax + by = c$ .
8. A debt of \$450 is to be shared equally among the members of the Outing Club. When five of the members refuse to pay, the other members' shares each go up by \$3. How many members does the Outing Club have?
9. Choose a point  $P$  on the line  $2x + 3y = 7$ , and draw the vector  $[2, 3]$  with its tail at  $P$  and its head at  $Q$ . Confirm that the vector is perpendicular to the line. What is the distance from  $Q$  to the line? Repeat the preceding, with a different choice for point  $P$ .
10. Let  $A = (3, 2)$  and  $B = (7, -10)$ . What is the displacement vector that moves point  $A$  onto point  $B$ ? What vector moves  $B$  onto  $A$ ?
11. Let  $M = (a, b)$ ,  $N = (c, d)$ ,  $M' = (a + h, b + k)$ , and  $N' = (c + h, d + k)$ . Show that segments  $MN$  and  $M'N'$  have the same length. Explain why this could be expected.

## Mathematics 2

1. The position of a bug is described by the parametric equation  $(x, y) = (2 - 12t, 1 + 5t)$ . Explain why the speed of the bug is 13 cm/sec. Change the equation to obtain the description of a bug moving along the same line with speed 26 cm/second.

2. Given the vector  $[-5, 12]$ , find the following vectors:

- (a) same direction, twice as long    (b) same direction, length 1  
(c) opposite direction, length 10    (d) opposite direction, length  $c$

3. *Some terminology:* When the components of the vector  $[5, -7]$  are multiplied by a given number  $t$ , the result may be written either as  $[5t, -7t]$  or as  $t[5, -7]$ . This is called the *scalar multiple* of vector  $[5, -7]$  by the *scalar*  $t$ . Find components for the following scalar multiples:

- (a)  $[12, -3]$  by scalar 5                      (b)  $[\sqrt{5}, \sqrt{10}]$  by scalar  $\sqrt{5}$   
(c)  $[-\frac{3}{4}, \frac{2}{3}]$  by scalar  $-\frac{1}{2} + \frac{2}{6}$             (d)  $[p, q]$  by scalar  $\frac{p}{q}$

4. Find the lengths of the following vectors:

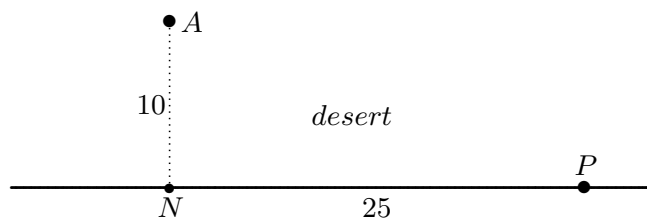
- (a)  $[3, 4]$                       (b)  $1998[3, 4]$             (c)  $\frac{1998}{5}[3, 4]$             (d)  $t[3, 4]$                       (e)  $t[a, b]$

5. With the aid of a ruler and protractor, draw and cut out three non-congruent triangles, each of which has a 40-degree angle, a 60-degree angle, and an 8-cm side. One of your triangles should have an angle-side-angle description, while the other two have angle-angle-side descriptions. What happens when you compare your triangles with those of your classmates?

6. A triangle has six principal parts — three sides and three angles. The SSS criterion states that three of these items (the sides) determine the other three (the angles). Are there other combinations of three parts that determine the remaining three? In other words, if the class is given three measurements with which to draw and cut out a triangle, which three measurements will guarantee that everyone's triangles will be congruent?

7. The initial position of an object is  $P_0 = (7, -2)$ . Its position after being displaced by the vector  $t[-8, 7]$  is  $P_t = (7, -2) + t[-8, 7]$ . Notice that the position is a function of  $t$ . Calculate  $P_3$ ,  $P_2$ , and  $P_{-2}$ . Describe the configuration of all possible positions  $P_t$ . By the way, “P sub t” and “P sub two” is how  $P_t$  and  $P_2$  are usually read.

8. Alex the geologist is in the desert, 10 km from a long, straight road. On the road, Alex's jeep can do 50 kph, but in the desert sands, it can only manage 30 kph. Alex is very thirsty, and knows that there is a gas station 25 km down the road (from the nearest point  $N$  on the road) that has ice-cold Pepsi.



- (a) How many minutes will it take for Alex to drive to  $P$  through the desert?  
(b) Would it be faster if Alex first drove to  $N$  and then used the road to  $P$ ?  
(c) Find an even faster route for Alex to follow. Is your route the fastest possible?



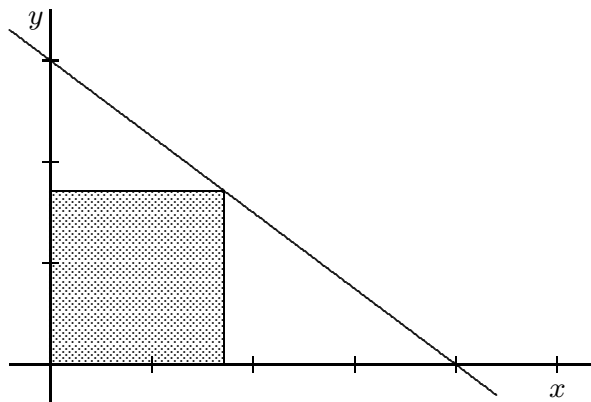
## Mathematics 2

1. Let  $A = (1, 4)$ ,  $B = (0, -9)$ ,  $C = (7, 2)$ , and  $D = (6, 9)$ . Prove that angles  $DAB$  and  $DCB$  are the same size. Can anything be said about the angles  $ABC$  and  $ADC$ ?

2. A puzzle: Cut out four copies of the quadrilateral  $ABCD$  formed by points  $A = (0, 0)$ ,  $B = (5, 0)$ ,  $C = (6, 2)$ , and  $D = (0, 5)$ . Show that it is possible to arrange these four pieces to form a square. Explain why you are sure that the pieces fit exactly.

3. Two of the sides of a right triangle have lengths  $360\sqrt{1994}$  and  $480\sqrt{1994}$ . Find the possible lengths for the third side.

4. The diagram at right shows the graph of  $3x + 4y = 12$ . The shaded figure is a square, three of whose vertices are on the coordinate axes. The fourth vertex is on the line. Find  
(a) the  $x$ - and  $y$ -intercepts of the line;  
(b) the length of a side of the square.



5. (Continuation) Draw a rectangle that is twice as wide as it is tall, and that fits snugly into the triangular region formed by the line  $3x + 4y = 12$  and the positive coordinate axes, with one corner at the origin and the opposite corner on the line. Find the dimensions of this rectangle.

6. Plot the three points  $P = (1, 3)$ ,  $Q = (5, 6)$ , and  $R = (11.4, 10.8)$ . Verify that  $PQ = 5$ ,  $QR = 8$ , and  $PR = 13$ . What is special about these points?

7. Sidney calculated three distances and reported them as  $TU = 29$ ,  $UV = 23$ , and  $TV = 54$ . What do you think of Sidney's data, and why?

8. Find the number that is two thirds of the way (a) from  $-7$  to  $17$ ; (b) from  $m$  to  $n$ .

9. The diagonal of a rectangle is 15 cm, and the perimeter is 38 cm. What is the area? It is possible to find the answer without finding the dimensions of the rectangle — try it.

10. After drawing the line  $y = 2x - 1$  and marking the point  $A = (-2, 7)$ , Kendall is trying to decide which point on the line is closest to  $A$ . The point  $P = (3, 5)$  looks promising. To check that  $P$  really is the point on  $y = 2x - 1$  that is closest to  $A$ , what should Kendall do? Is  $P$  closest to  $A$ ?

11. Dissect a 1-by-3 rectangle into three pieces that can be reassembled into a square.

12. Let  $K = (-2, 1)$  and  $M = (3, 4)$ . Find coordinates for the two points that divide segment  $KM$  into three congruent segments.

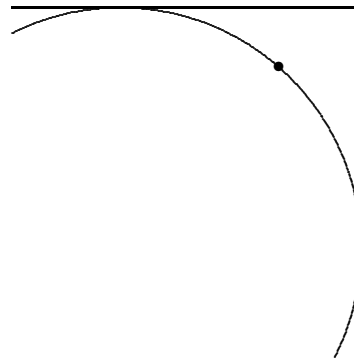
## Mathematics 2

1. The components of vector  $[24, 7]$  are 24 and 7. Find the components of a vector that is three fifths as long as  $[24, 7]$ .
2. Let  $A = (-5, 2)$  and  $B = (19, 9)$ . Find coordinates for the point  $P$  between  $A$  and  $B$  that is three fifths of the way from  $A$  to  $B$ . Find coordinates for the point  $Q$  between  $A$  and  $B$  that is three fifths of the way from  $B$  to  $A$ .
3. Given the points  $K = (-2, 1)$  and  $M = (3, 4)$ , find coordinates for a point  $J$  that makes angle  $JKM$  a right angle.
4. When two lines intersect, four angles are formed. It is not hard to believe that the nonadjacent angles in this arrangement are congruent. If you had to prove this to a skeptic, what reasons would you offer?
5. One of the legs of a right triangle is twice as long as the other, and the perimeter of the triangle is 28. Find the lengths of all three sides, to three decimal places.
6. A car traveling east at 45 miles per hour passes a certain intersection at 3 pm. Another car traveling north at 60 miles per hour passes the same intersection 25 minutes later. To the nearest minute, figure out when the cars are exactly 40 miles apart.
7. Find a point on the line  $y = 2x - 3$  that is 5 units from the  $x$ -axis.
8. Find a point on the line  $2x + y = 8$  that is equidistant from the coordinate axes. How many such points are there?
9. A line goes through the points  $(2, 5)$  and  $(6, -1)$ . Let  $P$  be the point on this line that is closest to the origin. Calculate the coordinates of  $P$ .
10. If I were to increase the length of my stride by one inch, it would take me 60 fewer strides to cover a mile. What was the length of my original stride?
11. The lines defined by  $P_t = (4 + 5t, -1 + 2t)$  and  $Q_u = (4 - 2u, -1 + 5u)$  intersect perpendicularly. Justify this statement. What are the coordinates of the point of intersection?
12. What number is exactly midway between  $23 - \sqrt{17}$  and  $23 + \sqrt{17}$ ? What number is exactly midway between  $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$  and  $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ ?
13. Given that  $P = (-1, -1)$ ,  $Q = (4, 3)$ ,  $A = (1, 2)$ , and  $B = (7, k)$ , find the value of  $k$  that makes the line  $AB$  (a) parallel to line  $PQ$ ; (b) perpendicular to line  $PQ$ .
14. Let  $A = (-6, -4)$ ,  $B = (1, -1)$ ,  $C = (0, -4)$ , and  $D = (-7, -7)$ . Show that the opposite sides of quadrilateral  $ABCD$  are parallel. A quadrilateral that has this property is called a *parallelogram*.

## Mathematics 2

1. Ashley saved a distance equal to 80% of the length of the shortest side of a rectangular field by cutting across the diagonal of the field instead of along two of the sides. Find the ratio of the length of the shortest side of the field to the length of its longest side.

2. A circular Harkness table is placed in a corner of a room so that it touches both walls. A mark is made on the edge of the table, exactly 18 inches from one wall and 25 inches from the other. What is the radius of the table?



3. If a line intersects the  $x$ -axis at  $(a, 0)$  and intersects the  $y$ -axis at  $(0, b)$ , at what point does it intersect the line  $y = x$ ?

4. Given  $A = (-1, 5)$ ,  $B = (x, 2)$ , and  $C = (4, -6)$  and the sum of  $AB + BC$  is to be a minimum, find the value of  $x$ .

5. On the same coordinate-axis system, graph the line defined by  $P_t = (3t - 4, 2t - 1)$  and the line defined by  $4x + 3y = 18$ . The graphs should intersect in the first quadrant.

(a) Calculate  $P_2$ , and show that it is not the point of intersection.

(b) Find the value of  $t$  for which  $P_t$  is on the line  $4x + 3y = 18$ .

6. The sides of a right triangle are  $x - y$ ,  $x$ , and  $x + y$ , where  $x$  and  $y$  are positive numbers, and  $y < x$ . Find the ratio of  $x$  to  $y$ .

7. After taking the Metro to Dupont Circle in Washington, DC, Jess reached street level by walking up the escalator at a brisk rate, taking 54 steps during the trip to the top. Suddenly curious about the length of the escalator, Jess returned to the bottom and walked up the same escalator at a leisurely rate, taking steps two thirds as often as on the first trip, taking 40 steps in all. How many steps can be seen on the visible part of the escalator?

8. Find a vector that translates the line  $2x - 3y = 18$  onto the line  $2x - 3y = 24$ . (There is more than one correct answer.)

9. Let  $A = (0, 0)$ ,  $B = (4, 2)$ , and  $C = (1, 3)$ , find the size of angle  $CAB$ . Justify your answer.

10. Let  $A = (3, 2)$ ,  $B = (1, 5)$ , and  $P = (x, y)$ . Find  $x$ - and  $y$ -values that make  $ABP$  a right angle.

11. (Continuation) Describe the configuration of all such points  $P$ .

12. Find coordinates for the vertices of a *lattice rectangle* that is three times as long as it is wide, with none of the sides horizontal.

## Mathematics 2

1. The vector that is defined by a directed segment  $AB$  is often denoted  $\overrightarrow{AB}$ . Find components for the following vectors  $\overrightarrow{AB}$ :

(a)  $A = (1, 2)$  and  $B = (3, -7)$       (b)  $A = (2, 3)$  and  $B = (2 + 3t, 3 - 4t)$

2. If  $A = (-2, 5)$  and  $B = (-3, 9)$ , find components for the vector that points

(a) from  $A$  to  $B$       (b) from  $B$  to  $A$

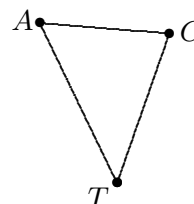
3. If  $M$  is the midpoint of segment  $AB$ , how are vectors  $\overrightarrow{AM}$ ,  $\overrightarrow{AB}$ ,  $\overrightarrow{MB}$ , and  $\overrightarrow{BM}$  related?

4. Choose positive integers  $m$  and  $n$ , with  $m < n$ . Let  $x = 2mn$ ,  $y = n^2 - m^2$ , and  $z = m^2 + n^2$ . It so happens that these three positive integers  $x$ ,  $y$ , and  $z$  have a special property. What is the property? Can you prove a general result?

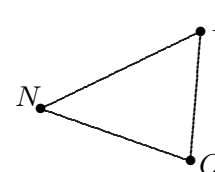
5. Show that the lines  $3x - 4y = -8$ ,  $x = 0$ ,  $3x - 4y = 12$ , and  $x = 4$  form the sides of a rhombus.

6. Suppose that triangle  $ACT$  has been shown to be congruent to triangle  $ION$ , with vertices  $A$ ,  $C$ , and  $T$  corresponding to vertices  $I$ ,  $O$ , and  $N$ , respectively. It is customary to record this result by writing  $\triangle ACT \cong \triangle ION$ . Notice that corresponding vertices occupy corresponding positions in the equation.

Let  $B = (5, 2)$ ,  $A = (-1, 3)$ ,  $G = (-5, -2)$ ,  $E = (1, -3)$ , and  $L = (0, 0)$ . Using only these five labels, find as many pairs of congruent triangles as you can, and express the congruences accurately.



7. (Continuation) How many ways are there of arranging the six letters of  $\triangle ACT \cong \triangle ION$  to express the two-triangle congruence?



8. What can be concluded about triangle  $ABC$  if it is given that

(a)  $\triangle ABC \cong \triangle ACB$       (b)  $\triangle ABC \cong \triangle BCA$

9. Plot points  $K = (-4, -3)$ ,  $L = (-3, 4)$ ,  $M = (-6, 3)$ ,  $X = (0, -5)$ ,  $Y = (6, -3)$ , and  $Z = (5, 0)$ . Show that triangle  $KLM$  is congruent to triangle  $XZY$ . Describe a transformation that transforms  $KLM$  onto  $XZY$ . Where does this transformation send the point  $(-5, 0)$ ?

10. Positions of three objects are described by the following three pairs of equations

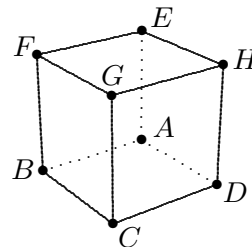
(a)  $\begin{cases} x = 2 - 2t \\ y = 5 + 7t \end{cases}$       (b)  $\begin{cases} x = 4 - 2t \\ y = -2 + 7t \end{cases}$       (c)  $\begin{cases} x = 2 - 2(t + 1) \\ y = 5 + 7(t + 1) \end{cases}$

How do the positions of these objects compare at any given moment?

11. Brooks and Avery are running laps around the outdoor track, in the same direction. Brooks completes a lap every 78 seconds while Avery needs 91 seconds for every tour of the track. Brooks (the faster runner) has just passed Avery. How much time will it take for Brooks to overtake Avery again?

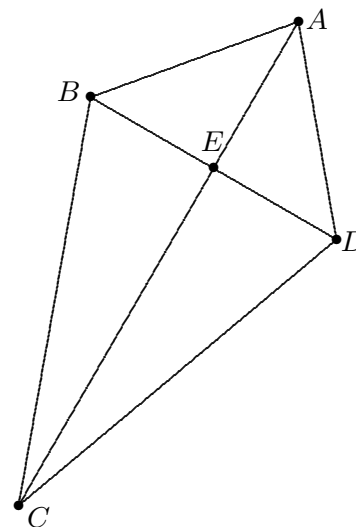
## Mathematics 2

1. Alex the geologist is in the desert, 10 km from the nearest point  $N$  on a long, straight road. Alex's jeep can do 50 kph on the road, and 30 kph in the desert. Find the shortest time for Alex to reach an oasis that is on the road **(a)** 20 km from  $N$ ; **(b)** 30 km from  $N$ .
2. Robin is moving on the coordinate plane according to the rule  $(x, y) = (-3 + 8t, 5 + 6t)$ , where distance is measured in km and time is measured in hours. Casey is following 20 km behind, at the same speed. Write parametric equations describing Casey's position.
3. Is it possible for a line to go through **(a)** *no* lattice points? **(b)** exactly *one* lattice point? **(c)** exactly *two* lattice points? For each answer, either give an example or else explain the impossibility.
4. Describe a transformation that carries the triangle with vertices  $(0, 0)$ ,  $(13, 0)$ , and  $(3, 2)$  onto the triangle with vertices  $(0, 0)$ ,  $(12, 5)$ , and  $(2, 3)$ . Where does your transformation send the point  $(6, 0)$ ?
5. Given  $A = (6, 1)$ ,  $B = (1, 3)$ , and  $C = (4, 3)$ , find a lattice point  $P$  that makes  $\overrightarrow{CP}$  perpendicular to  $\overrightarrow{AB}$ .
6. (Continuation) Describe the set of points  $P$  for which  $\overrightarrow{AB}$  and  $\overrightarrow{CP}$  are perpendicular.
7. The triangle with vertices  $(0, 0)$ ,  $(2, 1)$ , and  $(0, 5)$  can be cut into pieces that are each congruent to the triangle with vertices  $(2, 0)$ ,  $(3, 0)$ , and  $(3, 2)$ . Show how.
8. Let  $A = (0, 0)$ ,  $B = (1, 2)$ ,  $C = (6, 2)$ ,  $D = (2, -1)$ , and  $E = (1, -3)$ . Show that angle  $CAB$  is the same size as angle  $EAD$ .
9. Let  $A = (-2, 4)$  and  $B = (7, -2)$ . Find the point  $Q$  on the line  $y = 2$  that makes the total distance  $AQ + BQ$  as small as possible.
10. Let  $A = (-2, 4)$  and  $B = (7, 6)$ . Find the point  $P$  on the line  $y = 2$  that makes the total distance  $AP + BP$  as small as possible.
11. An ant is sitting at  $F$ , one of the eight vertices of a solid cube. It needs to crawl to vertex  $D$  as fast as possible. Find one of the shortest routes. How many are there?
12. A particle moves according to  $(x, y) = (6 - t, -1 + 3t)$ . For what value of  $t$  is the particle closest to the point  $(-2, 0)$ ?
13. Two automobiles each travel 60 km at steady rates. One car goes 6 kph faster than the other, thereby taking 20 minutes less time for the trip. Find the rate of the slower car.
14. What do the descriptions of position defined by equations  $P_t = (-2 + t, 3 + 2t)$  and  $Q_u = (4 + 3u, -1 + 6u)$  have in common? How do they differ?



## Mathematics 2

Here are some examples of proofs that do not use coordinates. They all show how specific *given* information can be used to logically deduce *new* information. Each example concerns a kite  $ABCD$ , for which  $AB = AD$  and  $BC = DC$  is the given information. The first two proofs show that diagonal  $AC$  creates angles  $BAC$  and  $DAC$  of the same size. The first proof consists of simple text; the second proof is written symbolically as an outline; this statement-reason form is sometimes called a “two-column proof.”



**Proof A:** Because  $AB = AD$  and  $BC = DC$ , and because the segment  $AC$  is shared by the triangles  $ABC$  and  $ADC$ , it follows from the SSS criterion that these triangles are congruent. Thus it is safe to assume that the corresponding parts of these triangles are also congruent (often abbreviated to CPCTC, as in proof 2 below). In particular, angles  $BAC$  and  $DAC$  are the same size.

<b>Proof B:</b>	$AB = AD$	given
	$BC = DC$	given
	$AC = AC$	shared side
	$\triangle ABC \cong \triangle ADC$	SSS
	$\angle BAC = \angle DAC$	CPCTC

1. In the fourth line, why would writing  $\triangle ABC \cong \triangle ACD$  have been incorrect?
2. Refer to the kite data above and prove that angles  $ABC$  and  $ADC$  are the same size.

Now let  $E$  be the intersection of diagonals  $AC$  and  $BD$ . The diagram makes it look like the diagonals intersect perpendicularly. Here are two proofs of this conjecture, each building on the result just proved.

**Proof C:** It is known that angles  $BAC$  and  $DAC$  are the same size (proof 1). Because  $AB = AD$  is given, and because edge  $AE$  is common to triangles  $BEA$  and  $DEA$ , it follows from the SAS criterion that these triangles are congruent. Their corresponding angles  $BEA$  and  $DEA$  must therefore be the same size. They are also supplementary, which makes them right angles, by definition.

<b>Proof D:</b>	$AB = AD$	given
	$\angle BAE = \angle DAE$	proof 2
	$AE = AE$	shared side
	$\triangle ABE \cong \triangle ADE$	SAS
	$\angle BEA = \angle DEA$	CPCTC
	$\angle BEA$ and $\angle DEA$ supplementary	$E$ is on $BD$
	$\angle BEA$ is right	definition of right angle

3. Using all of the above information, prove that  $AC$  bisects  $BD$ .

## Mathematics 2

1. An *altitude* of a triangle is a segment that joins one of the three vertices to a point on the line that contains the opposite side, the intersection being *perpendicular*. For example, consider the triangle whose vertices are  $A = (0, 0)$ ,  $B = (8, 0)$ , and  $C = (4, 12)$ .
  - (a) Find the length of the altitude from  $C$  to side  $AB$ . What is the area of  $ABC$ ?
  - (b) Find an equation for the line that contains the altitude from  $A$  to side  $BC$ .
  - (c) Find an equation for the line  $BC$ .
  - (d) Find coordinates for the point  $F$  where the altitude from  $A$  meets side  $BC$ . It is customary to call  $F$  the *foot* of the altitude from  $A$ .
  - (e) Find the length of the altitude from  $A$  to side  $BC$ .
  - (f) As a check on your work, calculate  $BC$  and multiply it by your answer to part (e). You should be able to predict the result.
  - (g) It is possible to deduce the length of the altitude from  $B$  to side  $AC$  from what you have already calculated. Show how.
2. If I were to increase my cycling speed by 3 mph, I calculate that it would take me 40 seconds less time to cover each mile. What is my current cycling speed?
3. Let  $A = (0, 0)$ ,  $B = (8, 1)$ ,  $C = (5, -5)$ ,  $P = (0, 3)$ ,  $Q = (7, 7)$ , and  $R = (1, 10)$ . Prove that angles  $ABC$  and  $PQR$  have the same size.
4. (Continuation) Let  $D$  be the point on segment  $AB$  that is exactly 3 units from  $B$ , and let  $T$  be the point on segment  $PQ$  that is exactly 3 units from  $Q$ . What evidence can you give for the congruence of triangles  $BCD$  and  $QRT$ ?
5. Find a point on the line  $x + 2y = 8$  that is equidistant from the points  $(3, 8)$  and  $(9, 6)$ .
6. Graph the line that is described parametrically by  $(x, y) = (2t, 5 - t)$ , then
  - (a) confirm that the point corresponding to  $t = 0$  is exactly 5 units from  $(3, 9)$ ;
  - (b) write a formula in terms of  $t$  for the distance from  $(3, 9)$  to  $(2t, 5 - t)$ ;
  - (c) find the other point on the line that is 5 units from  $(3, 9)$ ;
  - (d) find the point on the line that minimizes the distance to  $(3, 9)$ .
7. How large a square can be put inside a right triangle whose legs are 5 cm and 12 cm?
8. You are one mile from the railroad station, and your train is due to leave in ten minutes. You have been walking at a steady rate of 3 mph, and you can run at 8 mph if you have to. For how many more minutes can you continue walking, until it becomes necessary for you to run the rest of the way to the station?
9. If a quadrilateral is equilateral, its diagonals are perpendicular. True or false? Why?
10. The diagonals  $AC$  and  $BD$  of quadrilateral  $ABCD$  intersect at  $O$ . Given the information  $AO = BO$  and  $CO = DO$ , what can you deduce about the lengths of the sides of the quadrilateral? Prove your response.

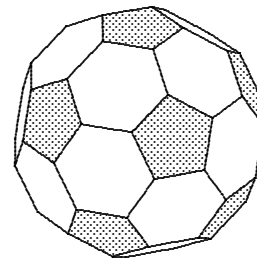
## Mathematics 2

1. Let  $A = (7, 7)$ ,  $B = (5, 1)$ , and  $P_t = (6 + 3t, 4 - t)$ . Plot  $A$  and  $B$ . Choose two values for  $t$  and plot the resulting points  $P_t$ , which should look equidistant from  $A$  and  $B$ . Make calculations to confirm the equidistance.
2. Make up a geometry problem to go with the equation  $x + 3x + x\sqrt{10} = 42$ .
3. Let  $A = (-2, 3)$ ,  $B = (6, 7)$ , and  $C = (-1, 6)$ .
  - (a) Find an equation for the perpendicular bisector of  $AB$ .
  - (b) Find an equation for the perpendicular bisector of  $BC$ .
  - (c) Find coordinates for a point  $K$  that is equidistant from  $A$ ,  $B$ , and  $C$ .
4. A segment from one of the vertices of a triangle to the midpoint of the opposite side is called a *median*. Consider the triangle defined by  $A = (-2, 0)$ ,  $B = (6, 0)$ , and  $C = (4, 6)$ .
  - (a) Find an equation for the line that contains the median drawn from  $A$  to  $BC$ .
  - (b) Find an equation for the line that contains the median drawn from  $B$  to  $AC$ .
  - (c) Find coordinates for  $G$ , the intersection of the medians from  $A$  and  $B$ .
  - (d) Let  $M$  be the midpoint of  $AB$ . Determine whether or not  $M$ ,  $G$ , and  $C$  are collinear.
5. The transformation defined by  $\mathcal{T}(x, y) = (y + 2, x - 2)$  is a reflection. Verify this by calculating the effect of  $\mathcal{T}$  on the triangle formed by  $P = (1, 3)$ ,  $Q = (2, 5)$ , and  $R = (6, 5)$ . Sketch triangle  $PQR$ , find coordinates for the *image points*  $P'$ ,  $Q'$ , and  $R'$ , and sketch the *image triangle*  $P'Q'R'$ . Then identify the mirror line and add it to your sketch. Notice that triangle  $PQR$  is labeled in a clockwise sense; what about the labels on triangle  $P'Q'R'$ ?
6. In quadrilateral  $ABCD$ , it is given that  $AB = CD$  and  $BC = DA$ . Prove that angles  $ACD$  and  $CAB$  are the same size. N.B. If a polygon has more than three vertices, the *labeling convention* is to place the letters around the polygon in the order that they are listed. Thus  $AC$  should be one of the diagonals of  $ABCD$ .
7. Maintaining constant speed and direction for an hour, Whitney traveled from  $(-2, 3)$  to  $(10, 8)$ . Where was Whitney after 35 minutes? What distance did Whitney cover in those 35 minutes?
8. A *direction vector* for a line is any vector that joins two points on that line. Find a direction vector for  $2x + 5y = 8$ . It is not certain that you and your classmates will get exactly the same answer. How should your answers be related, however?
9. (Continuation) Show that  $[b, -a]$  is a direction vector for the line  $ax + by = c$ .
10. (Continuation) Show that any direction vector for the line  $ax + by = c$  must be perpendicular to  $[a, b]$ .
11. A particle moves according to the equation  $(x, y) = (1, 2) + t[4, 3]$ . Let  $P$  be the point where the path of this particle intersects the line  $4x + 3y = 16$ . Find coordinates for  $P$ , then explain why  $P$  is the point on  $4x + 3y = 16$  that is closest to  $(1, 2)$ .



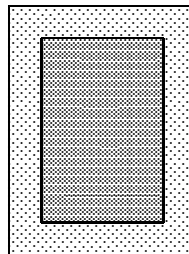
## Mathematics 2

1. True or false?  $\sqrt{4x} + \sqrt{9x} = \sqrt{13x}$
2. The line  $3x + 2y = 16$  is the perpendicular bisector of the segment  $AB$ . Find coordinates of point  $B$ , given that (a)  $A = (-1, 3)$ ; (b)  $A = (0, 3)$ .
3. (Continuation) Point  $B$  is called the *reflection of A across the line*  $3x + 2y = 16$ ; sometimes  $B$  is simply called the *image* of  $A$ . Explain this terminology. Using the same line, find another point  $C$  and its image  $C'$ . Explain your method for finding your pair of points.
4. A cube has 8 vertices, 12 edges, and 6 square faces. A soccer ball (also known as a *buckyball* or *truncated icosahedron*) has 12 pentagonal faces and 20 hexagonal faces. How many vertices and how many edges does a soccer ball have?
5. A rhombus has 25-cm sides, and one diagonal is 14 cm long. How long is the other diagonal?
6. Let  $A = (0, 0)$  and  $B = (12, 5)$ , and let  $C$  be the point on segment  $AB$  that is 8 units from  $A$ . Find coordinates for  $C$ .
7. Let  $A = (0, 0)$  and  $B = (12, 5)$ . There are points on the  $y$ -axis that are twice as far from  $B$  as they are from  $A$ . Make a diagram that shows these points, and use it to estimate their coordinates. Then use algebra to find them exactly.
8. Let  $A = (1, 4)$ ,  $B = (8, 0)$ , and  $C = (7, 8)$ . Find the area of triangle  $ABC$ .
9. Sketch triangle  $PQR$ , where  $P = (1, 1)$ ,  $Q = (1, 2)$ , and  $R = (3, 1)$ . For each of the following, apply the given transformation  $\mathcal{T}$  to the vertices of triangle  $PQR$ , sketch the image triangle  $P'Q'R'$ , then decide which of the terms *reflection*, *rotation*, *translation*, or *glide-reflection* accurately describes the action of  $\mathcal{T}$ . Provide appropriate detail to justify your choices.
  - (a)  $\mathcal{T}(x, y) = (x + 3, y - 2)$
  - (b)  $\mathcal{T}(x, y) = (y, x)$
  - (c)  $\mathcal{T}(x, y) = (-x + 2, -y + 4)$
  - (d)  $\mathcal{T}(x, y) = (x + 3, -y)$
10. Prove that one of the diagonals of a kite bisects two of the angles of the kite. What about the other diagonal — must it also be an *angle bisector*? Explain your response.
11. Let  $A = (2, 9)$ ,  $B = (6, 2)$ , and  $C = (10, 10)$ . Verify that segments  $AB$  and  $AC$  have the same length. Measure angles  $ABC$  and  $ACB$ . On the basis of your work, propose a general statement that applies to any triangle that has two sides of equal length. Prove your assertion, which might be called the *Isosceles-Triangle Theorem*.
12. If the diagonals of a quadrilateral bisect each other, then any two nonadjacent sides of the figure must have the same length. Prove that this is so.



## Mathematics 2

1. Robin is mowing a rectangular field that measures 24 yards by 32 yards, by pushing the mower around and around the outside of the plot. This creates a widening border that surrounds the unmowed grass in the center. During a brief rest, Robin wonders whether the job is half done yet. How wide is the uniform mowed border when Robin *is* half done?



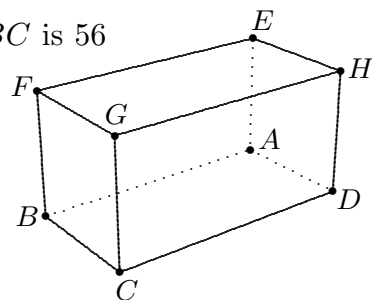
2. A geometric transformation is called an *isometry* if it preserves distances, in the following sense: The distance from  $M$  to  $N$  must be the same as the distance from  $M'$  to  $N'$ , for any two points  $M$  and  $N$  and their respective images  $M'$  and  $N'$ . You have already shown in a previous exercise that any translation is an isometry. Now let  $M = (a, b)$ ,  $N = (c, d)$ ,  $M' = (a, -b)$ , and  $N' = (c, -d)$ . Confirm that segments  $MN$  and  $M'N'$  have the same length, thereby showing that a certain transformation  $\mathcal{T}$  is an isometry. What type of transformation is  $\mathcal{T}$ ?

3. Use the distance formula to show that  $\mathcal{T}(x, y) = (-y, x)$  is an isometry.

4. Triangle  $ABC$  is isosceles, with  $AB = BC$ , and angle  $BAC$  is 56 degrees. Find the remaining two angles of this triangle.

5. Triangle  $ABC$  is isosceles, with  $AB = BC$ , and angle  $ABC$  is 56 degrees. Find the remaining two angles of this triangle.

6. An ant is sitting at  $F$ , one of the vertices of a solid rectangular block. Edges  $AD$  and  $AE$  are each half the length of edge  $AB$ . The ant needs to crawl to vertex  $D$  as fast as possible. Find one of the shortest routes. How many are there?



7. Suppose that vectors  $[a, b]$  and  $[c, d]$  are perpendicular. Show that  $ac + bd = 0$ .

8. Suppose that  $ac + bd = 0$ . Show that vectors  $[a, b]$  and  $[c, d]$  are perpendicular. The number  $ac + bd$  is called the *dot product* of the vectors  $[a, b]$  and  $[c, d]$ .

9. Let  $A = (0, 0)$ ,  $B = (4, 3)$ ,  $C = (2, 4)$ ,  $P = (0, 4)$ , and  $Q = (-2, 4)$ . Decide whether angles  $BAC$  and  $PAQ$  are the same size (congruent, that is), and give your reasons.

10. Let  $A = (-4, 0)$ ,  $B = (0, 6)$ , and  $C = (6, 0)$ .

(a) Find equations for the three lines that contain the altitudes of triangle  $ABC$ .

(b) Show that the three altitudes are *concurrent*, by finding coordinates for their common point. The point of concurrence is called the *orthocenter* of triangle  $ABC$ .

11. The equation  $y - 5 = m(x - 2)$  represents a line, no matter what value  $m$  has.

(a) What do all these lines have in common?

(b) When  $m = -2$ , what are the  $x$ - and  $y$ -intercepts of the line?

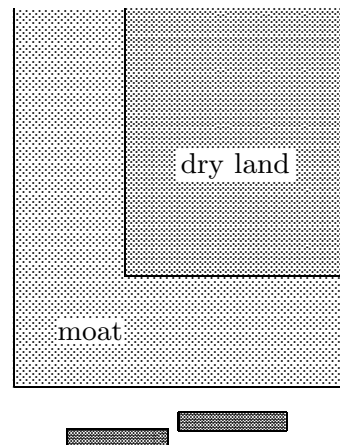
(c) When  $m = -1/3$ , what are the  $x$ - and  $y$ -intercepts of the line?

(d) When  $m = 2$ , what are the  $x$ - and  $y$ -intercepts of the line?

(e) For what values of  $m$  are the axis intercepts both positive?

## Mathematics 2

1. Find the area of the triangle whose vertices are  $A = (-2, 3)$ ,  $B = (6, 7)$ , and  $C = (0, 6)$ .
2. If triangle  $ABC$  is isosceles, with  $AB = AC$ , then the medians drawn from vertices  $B$  and  $C$  must have the same length. Write a *two-column proof* of this result.
3. Let  $A = (-4, 0)$ ,  $B = (0, 6)$ , and  $C = (6, 0)$ .
  - (a) Find equations for the three medians of triangle  $ABC$ .
  - (b) Show that the three medians are concurrent, by finding coordinates for their common point. The point of concurrence is called the *centroid* of triangle  $ABC$ .
4. Given points  $A = (0, 0)$  and  $B = (-2, 7)$ , find coordinates for points  $C$  and  $D$  so that  $ABCD$  is a square.
5. Given the transformation  $\mathcal{F}(x, y) = (-0.6x - 0.8y, 0.8x - 0.6y)$ , Shane calculated the image of the isosceles right triangle formed by  $S = (0, 0)$ ,  $H = (0, -5)$ , and  $A = (5, 0)$ , and declared that  $\mathcal{F}$  is a reflection. Morgan instead calculated the image of the *scalene* (non-isosceles) triangle formed by  $M = (7, 4)$ ,  $O = (0, 0)$ , and  $R = (7, 1)$ , and concluded that  $\mathcal{F}$  is a rotation. Who was correct? Explain your choice, and account for the disagreement.
6. Let  $A = (0, 12)$  and  $B = (25, 12)$ . If possible, find coordinates for a point  $P$  on the  $x$ -axis that makes angle  $APB$  a right angle.
7. Brett and Jordan are out driving in the coordinate plane, each on a separate straight road. The equations  $B_t = (-3, 4) + t[1, 2]$  and  $J_t = (5, 2) + t[-1, 1]$  describe their respective travels, where  $t$  is the number of minutes after noon.
  - (a) Make a sketch of the two roads, with arrows to indicate direction of travel.
  - (b) Where do the two roads intersect?
  - (c) How fast is Brett going? How fast is Jordan going?
  - (d) Do they collide? If not, who gets to the intersection first?
8. A castle is surrounded by a rectangular moat, which is of uniform width 12 feet. A corner is shown in the top view at right. The problem is to get across the moat to dry land on the other side, without using the drawbridge. To work with, you have only two rectangular planks, whose lengths are 11 feet and 11 feet, 9 inches. Show how the planks can get you across.
9. Find  $k$  so that the vectors  $[4, -3]$  and  $[k, -6]$ 
  - (a) point in the same direction;
  - (b) are perpendicular.
10. The lines  $3x + 4y = 12$  and  $3x + 4y = 72$  are parallel. Explain why. Find the distance that separates these lines. You will have to decide what “distance” means in this context.
11. Give an example of an equiangular polygon that is not equilateral.
12. Find coordinates for a point on the line  $4y = 3x$  that is 8 units from  $(0, 0)$ .



## Mathematics 2

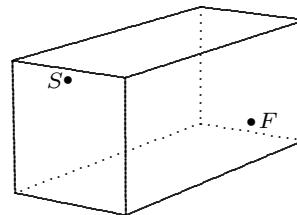
1. An object moves with constant *velocity* (which means constant speed and direction) from  $(-3, 1)$  to  $(5, 7)$ , taking five seconds for the trip.

(a) What is the speed of the object?

(b) Where does the object cross the  $y$ -axis?

(c) Where is the object three seconds after it leaves  $(-3, 1)$ ?

2. A spider lived in a room that measured 30 feet long by 12 feet wide by 12 feet high. One day, the spider spied an incapacitated fly across the room, and of course wanted to crawl to it as quickly as possible. The spider was on an end wall, one foot from the ceiling and six feet from each of the long walls. The fly was stuck one foot from the floor on the opposite wall, also midway between the two long walls. Knowing some geometry, the spider cleverly took the shortest possible route to the fly and ate it for lunch. How far did the spider crawl?



3. Told to investigate the transformation  $\mathcal{T}(x, y) = (x + 3, 2y + 1)$ , Morgan calculated the images of  $P = (1, 5)$  and  $Q = (-3, 5)$ . Because  $PQ$  and  $P'Q'$  are equal, Morgan declared that  $\mathcal{T}$  is an isometry. Shane disagreed with this conclusion. Who is correct, and why?

4. Find the area of the parallelogram whose vertices are  $(0, 0)$ ,  $(7, 2)$ ,  $(8, 5)$ , and  $(1, 3)$ .

5. Find the point on the  $y$ -axis that is equidistant from  $A = (0, 0)$  and  $B = (12, 5)$ .

6. Given the points  $A = (0, 0)$ ,  $B = (7, 1)$ , and  $D = (3, 4)$ , find coordinates for the point  $C$  that makes quadrilateral  $ABCD$  a parallelogram. What if the question had requested  $ABDC$  instead?

7. Find a vector that is perpendicular to the line  $3x - 4y = 6$ .

8. Measurements are made on quadrilaterals  $ABCD$  and  $PQRS$ , and it is found that angles  $A$ ,  $B$ , and  $C$  are the same size as angles  $P$ ,  $Q$ , and  $R$ , respectively, and that sides  $AB$  and  $BC$  are the same length as  $PQ$  and  $QR$ , respectively. Is this enough evidence to conclude that the quadrilaterals  $ABCD$  and  $PQRS$  are congruent? Explain.

9. Let  $P = (-1, 3)$ . Find the point  $Q$  for which the line  $2x + y = 5$  serves as the perpendicular bisector of segment  $PQ$ .

10. Let  $A = (3, 4)$ ,  $B = (0, -5)$ , and  $C = (4, -3)$ . Find equations for the perpendicular bisectors of segments  $AB$  and  $BC$ , and coordinates for their common point  $K$ . Calculate lengths  $KA$ ,  $KB$ , and  $KC$ . Why is  $K$  also on the perpendicular bisector of segment  $CA$ ?

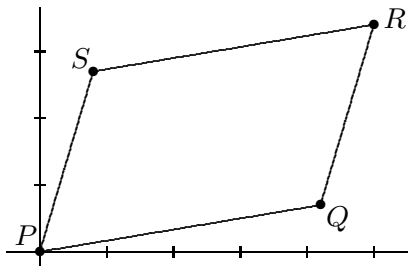
11. (Continuation) A circle centered at  $K$  can be drawn so that it goes through all three vertices of triangle  $ABC$ . Explain. This is why  $K$  is called the *circumcenter* of the triangle. In general, how do you locate the circumcenter of a triangle?

## Mathematics 2

1. The equation  $y - 5 = m(x - 2)$  represents a line, no matter what value  $m$  has.
  - (a) What are the  $x$ - and  $y$ -intercepts of this line?
  - (b) For what value of  $m$  does this line form a triangle of area 36 with the positive axes?
  - (c) Show that the area of a first-quadrant triangle is always at least 20.

2. The figure at right shows a parallelogram  $PQRS$ , three of whose vertices are  $P = (0, 0)$ ,  $Q = (a, b)$ , and  $S = (c, d)$ .

- (a) Find the coordinates of  $R$ .
  - (b) Find the area of  $PQRS$ , and simplify your formula.



3. Working against a 1-km-per-hour current, some members of the Outing Club paddled 7 km up the Exeter River one Saturday last spring and made camp. The next day, they returned downstream to their starting point, aided by the same one-km-per-hour current. They paddled for a total of 6 hours and 40 minutes during the round trip. Use this information to figure out how much time the group would have needed to make the trip if there had been no current.

4. Decide whether the transformation  $\mathcal{T}(x, y) = (2x, \frac{1}{2}y)$  is an isometry, and give your reasons.

5. Find points on the line  $3x + 5y = 15$  that are equidistant from the coordinate axes.

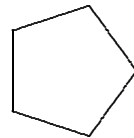
6. Plot all points that are 3 units from the  $x$ -axis. Describe the configuration.

7. Plot all points that are 3 units from the  $x$ -axis and 3 units from  $(5, 4)$ . How many did you find?

8. In triangle  $ABC$ , it is given that  $CA = CB$ . Points  $P$  and  $Q$  are marked on segments  $CA$  and  $CB$ , respectively, so that angles  $CBP$  and  $CAQ$  are the same size. Prove that  $CP = CQ$ .

9. (Continuation) Segments  $BP$  and  $AQ$  intersect at  $K$ . Explain why you can be sure that quadrilateral  $CPKQ$  is a kite. You might want to consider triangles  $AKP$  and  $BKQ$ .

10. A polygon that is both equilateral and equiangular is called *regular*. Prove that all diagonals of a regular *pentagon* (five sides) have the same length.



11. Find coordinates for the point equidistant from  $(-1, 5)$ ,  $(8, 2)$ , and  $(6, -2)$ .

12. Find coordinates for the point where line  $(x, y) = (3 + 2t, -1 + 3t)$  meets line  $y = 2x - 5$ .

13. Find an equation for the line that goes through  $(5, 2)$  and that forms a triangle in the first quadrant that is just large enough to enclose the 4-by-4 square in the first quadrant that has two of its sides on the coordinate axes.

14. Find the area of the parallelogram whose vertices are  $(2, 5)$ ,  $(7, 6)$ ,  $(10, 10)$ , and  $(5, 9)$ .

## Mathematics 2

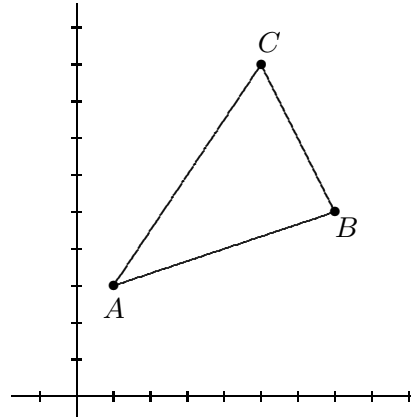
1. Let  $E = (2, 7)$  and  $F = (10, 1)$ . On the line through  $E$  and  $F$ , there are two points that are 3 units from  $E$ . Find coordinates for both of them.

2. Let  $A = (1, 3)$ ,  $B = (7, 5)$ , and  $C = (5, 9)$ . Answer the item below that is determined by *the first letter of your last name*. Find coordinates for the requested point.

(a-e) Show that the three medians of triangle  $ABC$  are concurrent at a point  $G$ .

(f-m) Show that the three altitudes of triangle  $ABC$  are concurrent at a point  $H$ .

(n-z) Show that the perpendicular bisectors of the sides of triangle  $ABC$  are concurrent at a point  $K$ . What special property does  $K$  have?



3. (Continuation for class discussion) It looks like  $G$ ,  $H$ , and  $K$  are collinear. Are they?

4. Show that the following transformations are isometries, and identify their type:

(a)  $\mathcal{T}(x, y) = (-x, y + 2)$

(b)  $\mathcal{T}(x, y) = (0.6x - 0.8y, 0.8x + 0.6y)$

5. Find coordinates for a point that is three times as far from the origin as  $(2, 3)$  is. Describe the configuration of all such points.

6. What are the axis intercepts of the line described by  $P_t = (5 + 3t, -2 + 4t)$ ?

7. Simplify equation  $\sqrt{(x - 3)^2 + (y - 5)^2} = \sqrt{(x - 7)^2 + (y + 1)^2}$ . Interpret your result.

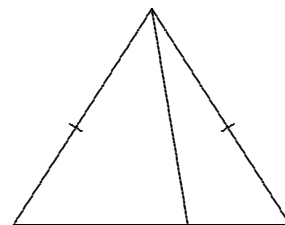
8. How large an equilateral triangle can you fit inside a 2-by-2 square?

9. Plot the points  $K = (0, 0)$ ,  $L = (7, -1)$ ,  $M = (9, 3)$ ,  $P = (6, 7)$ ,  $Q = (10, 5)$ , and  $R = (1, 2)$ . You will see that the triangles  $KLM$  and  $RPQ$  are congruent. Find coordinates for the point in triangle  $KLM$  that corresponds to  $(3, 4)$  in triangle  $RPQ$ .

10. Given that  $ABCDEFGHI$  is a regular polygon, prove that  $AD$  and  $FI$  have the same length.

11. Find a fourth-quadrant point that is equally distant from  $(4, 1)$  and the  $y$ -axis.

12. Use the diagram to help you explain why SSA evidence is not by itself sufficient to justify the congruence of triangles. The tick marks designate segments that have the same length.



13. The diagonals of a kite are 6 cm and 12 cm long. Is it possible for the lengths of the sides of this kite to be in a 2-to-1 ratio?

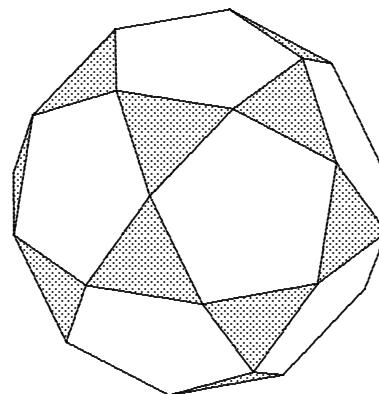
14. Translate the line  $5x + 7y = 35$  by vector  $[3, 10]$ . Find an equation for the new line.

## Mathematics 2

1. Let  $A = (\sqrt{15}, \sqrt{11})$ ,  $B = (\sqrt{11}, -\sqrt{15})$ ,  $C = (1, 2)$ , and  $D = (7, 6)$ . Is there an isometry that transforms segment  $AB$  onto segment  $CD$ ? Explain.

2. Consider the following process for bisecting an angle  $ABC$ : First mark  $M$  on  $BA$  and  $P$  on  $BC$  so that  $MB = PB$ , then mark new points  $N$  on  $BA$  and  $Q$  on  $BC$  so that  $NB = QB$ . Let  $E$  be the intersection of  $MQ$  and  $NP$ . Prove that segment  $BE$  is the desired angle bisector.

3. You have recently seen that there is no completely reliable SSA criterion for congruence. If the angle part of such a correspondence is a *right* angle, however, the criterion *is* reliable. Justify this so-called *hypotenuse-leg* criterion (which is abbreviated HL).



4. An *icosidodecahedron* has twelve pentagonal faces, as shown at right. How many edges does this figure have? How many vertices? How many triangular faces?

5. It is given that  $a + b = 6$  and  $ab = 7$ .

(a) Find the value of  $a^2 + b^2$ . Can you do this without finding values for  $a$  and  $b$ ?

(b) Make up a geometry word problem that corresponds to the question in part (a).

6. Avery can run at 10 uph. The bank of a river is represented by the line  $4x + 3y = 12$ , and Avery is at  $(7, 5)$ . How much time does Avery need to reach the river?

7. Find an equation for the line through point  $(7, 9)$  that is perpendicular to vector  $[5, -2]$ .

8. Describe a transformation that carries the triangle with vertices  $R = (1, 2)$ ,  $P = (6, 7)$ , and  $Q = (10, 5)$  onto the triangle with vertices  $K = (0, 0)$ ,  $L = (7, -1)$ , and  $M = (9, 3)$ . Where does your transformation send (a)  $(4, 5)$ ? (b)  $(7, 5)$ ?

9. If the parts of two triangles are matched so that two angles of one triangle are congruent to the corresponding angles of the other, and so that a side of one triangle is congruent to the corresponding side of the other, then the triangles must be congruent. Justify this *angle-angle-corresponding side* (AAS) criterion for congruence. Would AAS be a valid test for congruence if the word *corresponding* were left out of the definition? Explain.

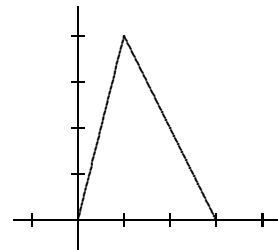
10. Suppose that triangle  $PAB$  is isosceles, with  $AP = PB$ , and that  $C$  is on side  $PB$ , between  $P$  and  $B$ . Show that  $CB < AC$ .

11. Apply the transformation  $\mathcal{T}(x, y) = \left( \frac{\sqrt{3}}{2}x + \frac{1}{2}y, -\frac{1}{2}x + \frac{\sqrt{3}}{2}y \right)$  to the triangle whose vertices are  $(0, 0)$ ,  $(4, 0)$ , and  $(0, 8)$ . Is  $\mathcal{T}$  an isometry?

12. A triangle that has a 13-inch side, a 14-inch side, and a 15-inch side has an area of 84 square inches. Accepting this fact, find the lengths of all three altitudes of this triangle.

## Mathematics 2

1. Draw the lines  $y = 0$ ,  $y = \frac{1}{2}x$ , and  $y = \frac{4}{3}x$ . Use your protractor to measure the angles, then make calculations to confirm what you observe.
2. Find the point of intersection of the lines  $P_t = (-1 + 3t, 3 + 2t)$  and  $Q_r = (4 - r, 1 + 2r)$ .
3. Find the area of the triangle having sides 10, 10, and 5.
4. Apply the transformation  $\mathcal{T}(x, y) = (2x + 3y, -x + y)$  to the unit square, whose vertices are  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ . Even though  $\mathcal{T}$  is not a reflection, it is customary to call the resulting figure the *image* of the square. What kind of figure is it?
5. Explain why an isometry always transforms a right triangle onto a right triangle.
6. (Continuation) Consider a transformation  $\mathcal{T}$  for which the image of the  $x$ -axis is the line  $2x + 3y = 6$  and the image of the  $y$ -axis is the line  $x + 7y = 7$ . What must be the image of the origin? Could  $\mathcal{T}$  be an isometry?
7. (Continuation) Consider a transformation  $\mathcal{T}$  for which the image of the  $x$ -axis is the line  $3x - 2y = 12$  and the image of the  $y$ -axis is the line  $2x + 3y = 6$ . What must be the image of the origin? Could  $\mathcal{T}$  be an isometry?
8. Find the lengths of *all* the altitudes of the triangle whose vertices are  $(0, 0)$ ,  $(3, 0)$ , and  $(1, 4)$ .
9. Form a triangle using three lattice points of your choosing. Verify that the medians of your triangle are concurrent.
10. Let  $P = (2, 7)$ ,  $B = (6, 11)$ , and  $M = (5, 2)$ . Find a point  $D$  that makes  $\overrightarrow{PB} = \overrightarrow{DM}$ . What can you say about quadrilateral  $PBMD$ ?
11. When translation by vector  $[2, 5]$  is followed by translation by vector  $[5, 7]$ , the net result can be achieved by applying a *single* translation; what is its vector?
12. Given that  $(-1, 4)$  is the reflected image of  $(5, 2)$ , find an equation for the line of reflection.
13. Draw a parallelogram whose adjacent edges are determined by vectors  $[2, 5]$  and  $[7, -1]$ , placed so that they have a common initial point. This is called placing vectors *tail-to-tail*. Find the area of the parallelogram.
14. Point  $(0, 1)$  is reflected across the line  $2x + 3y = 6$ . Find coordinates for its image.
15. A stop sign — a regular *octagon* — can be formed from a 12-inch square sheet of metal by making four straight cuts that snip off the corners. How long, to the nearest 0.01 inch, are the sides of the resulting polygon?





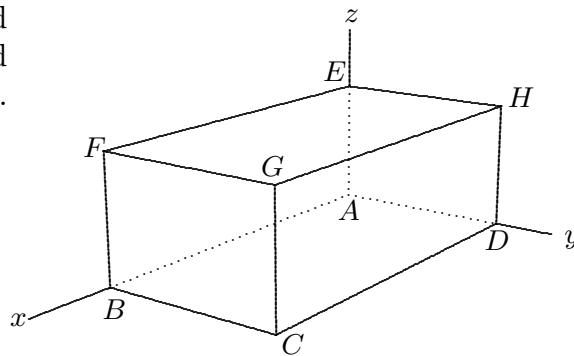
## Mathematics 2

1. The diagram shows a rectangular box named  $ABCDEFGH$ . Notice that  $A = (0, 0, 0)$ , and that  $B$ ,  $D$ , and  $E$  are on the coordinate axes.

Given that  $G = (6, 3, 2)$ , find

(a) coordinates for the other six vertices;

(b) the lengths  $AH$ ,  $AC$ ,  $AF$ , and  $AG$ .



2. The edges of a rectangular solid are parallel to the coordinate axes, and it has the points  $(0, 0, 0)$  and  $(8, 4, 4)$  as diagonally opposite vertices. Make a sketch, labeling each vertex with its coordinates, then find (a) the distance from  $(8, 4, 4)$  to  $(0, 0, 0)$  and (b) the distance from  $(8, 4, 4)$  to the  $z$ -axis.

3. Find components for the vector that points from  $(1, 1, 1)$  to  $(2, 3, 4)$ . Then find the distance from  $(1, 1, 1)$  to  $(2, 3, 4)$  by finding the length of this vector.

4. There are many points in the first quadrant (of the  $xy$ -plane) that are the same distance from the  $x$ -axis as they are from the point  $(0, 2)$ . Make a sketch that shows several of them. Use your ruler to check that the points you plotted satisfy the required equidistance property.

5. Find the distance from the origin to (a)  $(3, -2, 2)$ ; (b)  $(a, b, c)$ .

6. Find the area of a triangle formed by placing the vectors  $[3, 6]$  and  $[7, 1]$  tail-to-tail.

7. (Continuation) Describe your triangle using a different pair of vectors.

8. (Continuation) Find the length of the longest altitude of your triangle.

9. The diagonals of quadrilateral  $ABCD$  intersect perpendicularly at  $O$ . What can be said about quadrilateral  $ABCD$ ?

10. The *sum* of two vectors  $[a, b]$  and  $[p, q]$  is defined as  $[a + p, b + q]$ . Find the components of the vectors (a)  $[2, 3] + [-7, 5]$ ; (b)  $[3, 4, 8] + [4, 2, 5]$ .

11. Find  $x$  so that the distance from  $(x, 3, 6)$  to the origin is 9 units.

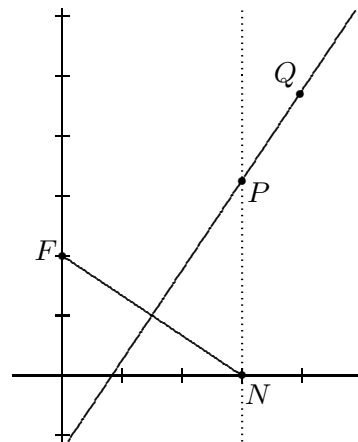
12. Write an equation that says the point  $(x, y, z)$  is 3 units from the origin. Describe the configuration of all such points.

13. Choose coordinates for three non-collinear points  $A$ ,  $B$ , and  $C$ . Calculate components for the vectors  $\overrightarrow{AB}$ ,  $\overrightarrow{AC}$ , and  $\overrightarrow{AB} + \overrightarrow{AC}$ , and then translate point  $A$  by vector  $\overrightarrow{AB} + \overrightarrow{AC}$ . Call the new point  $D$ . What kind of quadrilateral is  $ABDC$ ?

14. What do you call (a) an *equiangular quadrilateral*? (b) an *equilateral quadrilateral*?

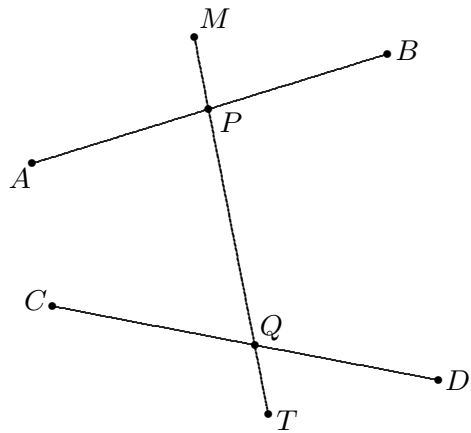
## Mathematics 2

1. Let  $A = (5, -3, 6)$ ,  $B = (0, 0, 0)$ , and  $C = (3, 7, 1)$ . Show that  $ABC$  is a right angle.
2. Find components for a vector of length 21 that points in the same direction as  $[2, 3, 6]$ .
3. Find coordinates for the point on segment  $KL$  that is 5 units from  $K$ , where  
**(a)**  $K = (0, 0, 0)$  and  $L = (4, 7, 4)$ ;      **(b)**  $K = (3, 2, 1)$  and  $L = (7, 9, 5)$ .
4. Simplify the sum of vectors  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ .
5. In quadrilateral  $ABCD$ , it is given that  $\overrightarrow{AB} = \overrightarrow{DC}$ . What kind of a quadrilateral is  $ABCD$ ? What can be said about the vectors  $\overrightarrow{AD}$  and  $\overrightarrow{BC}$ ?
6. Mark a lattice point on your graph paper. Define vector  $\mathbf{u}$  (which can be handwritten  $\overrightarrow{u}$ ) by moving 5 units to the right and 2 units up. Define vector  $\mathbf{v}$  by moving 1 unit to the right and 3 units down. Diagram the vectors  $\mathbf{u} + \mathbf{v}$ ,  $\mathbf{u} - \mathbf{v}$ ,  $\mathbf{u} + 2\mathbf{v}$ , and  $2\mathbf{u} - 3\mathbf{v}$ .
7. Draw a parallelogram. Choose one of its vertices and let  $\mathbf{u}$  and  $\mathbf{v}$  be the vectors defined by the sides that originate at that vertex. Draw  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$ . The vectors  $\mathbf{u}$  and  $\mathbf{v}$  represent the sides of the parallelogram; what do  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$  represent?
8. Let  $F = (0, 2)$  and  $N = (3, 0)$ . Find coordinates for the point  $P$  where the perpendicular bisector of segment  $FN$  intersects the line  $x = 3$ .
9. (Continuation) Choose a new  $N \neq (3, 0)$  on the  $x$ -axis, and repeat the calculation of  $P$ : Find the intersection of the perpendicular bisector of  $FN$  with the line through  $N$  that is parallel to the  $y$ -axis. Explain why there is always an intersection point  $P$ , no matter what  $N$  is chosen.
10. (Continuation) Given  $Q \neq P$  on the perpendicular bisector of  $FN$ , show that the distance from  $Q$  to  $F$  exceeds the distance from  $Q$  to the  $x$ -axis. Why was it necessary to exclude the case  $Q = P$ ?
11. The edges of a rectangular solid are parallel to the coordinate axes, and it has the points  $(2, 4, 4)$  and  $(6, 9, 1)$  as diagonally opposite vertices. Make a sketch, labeling each vertex with its coordinates, then find **(a)** the dimensions of the solid and **(b)** the length of its diagonal.
12. Show that the vectors  $[5, -3, 6]$  and  $[3, 7, 1]$  are perpendicular.
13. Asked to reflect the point  $P = (4, 0)$  across the mirror line  $y = 2x$ , Aubrey reasoned this way: First mark the point  $A = (1, 2)$  on the line, then use the vector  $[-3, 2]$  from  $P$  to  $A$  to reach from  $A$  to  $P' = (-2, 4)$ , which is the requested image. Does this make sense to you? Explain.



## Mathematics 2

1. The diagram at right shows lines  $APB$  and  $CQD$  intersected by line  $MPQT$ , which is called a *transversal*. There are two groups of angles: one group of four angles with vertex at  $P$ , and another group with vertex at  $Q$ . There is special terminology to describe pairs of angles, *one from each group*. If the angles are on different sides of the transversal, they are called *alternate*, for example  $APM$  and  $PQD$ . Angle  $BPQ$  is an *interior* angle because it is between the lines  $AB$  and  $CD$ , and angle  $CQT$  is *exterior*. Thus angles  $APQ$  and  $PQD$  are called *alternate interior*, while angles  $MPB$  and  $CQT$  are called *alternate exterior*. On the other hand, the pair of angles  $MPB$  and  $PQD$  — which are non-alternate angles, one interior, and the other exterior — is called *corresponding*. Refer to the diagram and name



- (a) the other pair of alternate interior angles;
- (b) the other pair of alternate exterior angles;
- (c) the angles that correspond to  $CQT$  and to  $TQD$ .

2. Mark points  $A = (1, 7)$  and  $B = (6, 4)$  on your graph paper. Use your protractor to draw two lines of positive slope that make 40-degree angles with line  $AB$  — one through  $A$  and one through  $B$ . What can you say about these two lines, and how can you be sure?

3. If one pair of alternate interior angles is equal, what can you say about the two lines that are crossed by the transversal? If one pair of corresponding angles is equal, what can you say about the two lines that are crossed by the transversal?

4. If it is known that one pair of alternate interior angles is equal, what can be said about  
 (a) the other pair of alternate interior angles?    (b) either pair of alternate exterior angles?  
 (c) any pair of corresponding angles?    (d) either pair of non-alternate interior angles?

5. You probably know that the sum of the angles of a triangle is a straight angle. One way to confirm this is to draw a line through one of the vertices, parallel to the opposite side. This creates some alternate interior angles. Finish the demonstration.

6. Suppose that two of the angles of triangle  $ABC$  are known to be congruent to two of the angles of triangle  $PQR$ . What can be said about the third angles?

7. Suppose that  $ABCD$  is a square, and that  $CDP$  is an equilateral triangle, with  $P$  *outside* the square. What is the size of angle  $PAD$ ?

8. Write an equation that says that points  $(0, 0, 0)$ ,  $(a, b, c)$ , and  $(m, n, p)$  form a right triangle, the right angle being *at the origin*. Simplify your equation as much as you can.

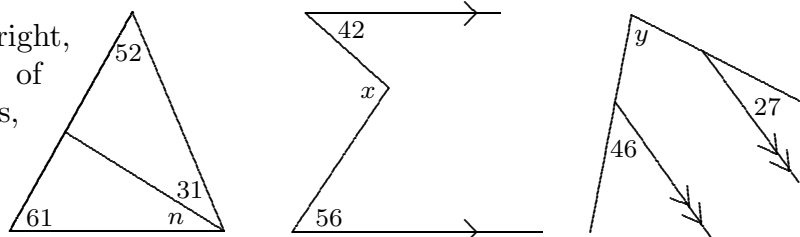
9. Write an equation that says that vectors  $[a, b, c]$  and  $[m, n, p]$  are perpendicular.

10. Write an equation that says that vectors  $[a, b]$  and  $[m, n]$  are perpendicular.

## Mathematics 2

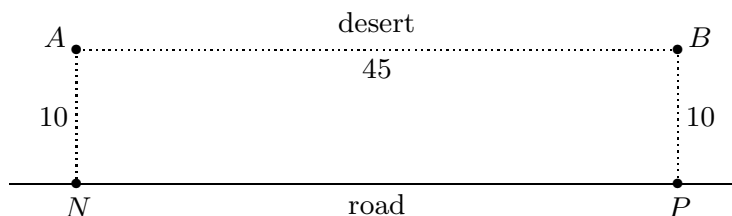
1. Give an example of a nonzero vector that is perpendicular to  $[5, 7, 4]$ .
2. Triangle  $ABC$  is isosceles, with  $AB$  congruent to  $AC$ . *Extend* segment  $BA$  to a point  $T$  (in other words,  $A$  should be between  $B$  and  $T$ ). Prove that angle  $TAC$  must be twice the size of angle  $ABC$ . Angle  $TAC$  is called one of the *exterior angles* of triangle  $ABC$ .
3. If  $ABC$  is any triangle, and  $TAC$  is one of its exterior angles, then what can be said about the size of angle  $TAC$ , in relation to the other angles of the figure?
4. Given triangle  $ABC$ , with  $AB = AC$ , extend segment  $AB$  to a point  $P$  so that  $BP = BC$ . In the resulting triangle  $APC$ , show that angle  $ACP$  is exactly three times the size of angle  $APC$ . (By the way, notice that extending segment  $AB$  does *not* mean the same thing as extending segment  $BA$ .)
5. Given an arbitrary triangle, what can you say about the *sum* of the three exterior angles, one for each vertex of the triangle?

6. In the diagrams shown at right, the goal is to find the sizes of the angles marked with letters, using the given numerical information. Angles are measured in degrees. Notice the custom of marking arrows on lines to indicate that they are known to be parallel.



7. Triangle  $ABC$  has a 34-degree angle at  $A$ . The bisectors of angles  $B$  and  $C$  meet at point  $I$ . What is the size of angle  $BIC$ ? Answer this question (a) assuming that  $ABC$  is right; (b) assuming that  $ABC$  is isosceles; (c) choosing sizes for angles  $B$  and  $C$ . Hmm...
8. Recall that a quadrilateral that has two pairs of parallel opposite sides is called a *parallelogram*. What can be said about the angles of such a figure?
9. Let  $ABCD$  be a parallelogram. (a) Express  $\overrightarrow{AC}$  in terms of  $\overrightarrow{AB}$  and  $\overrightarrow{BC}$ . (b) Express  $\overrightarrow{AC}$  in terms of  $\overrightarrow{AB}$  and  $\overrightarrow{AD}$ . (c) Express  $\overrightarrow{BD}$  in terms of  $\overrightarrow{AB}$  and  $\overrightarrow{AD}$ .

10. Alex the geologist is in the desert again, 10 km from a long, straight road and 45 km from base camp. The base camp is also 10 km from the road, on the same side of the road as Alex is.



On the road, the jeep can do 50 kph, but in the desert sands, it can only manage 30 kph. Alex wants to return to base camp as quickly as possible. How long will the trip take?

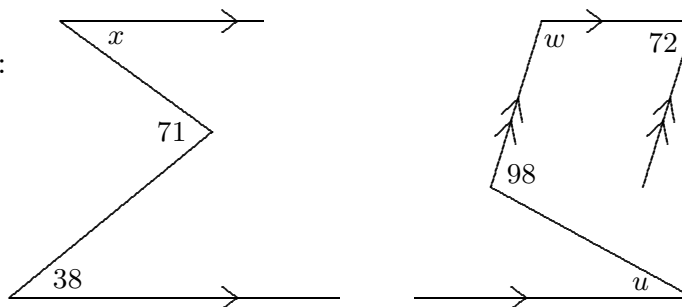
## Mathematics 2

1. Find an equation that says that  $P = (x, y)$  is equidistant from  $F = (2, 0)$  and the  $y$ -axis. Plot four points that fit this equation. The configuration of all such points  $P$  is called a *parabola*.

2. Prove that the sum of the angles of any quadrilateral is 360 degrees. What about the sum of the angles of a pentagon? a hexagon? a 57-sided polygon?

3. Sketch the rectangular box that has one corner at  $A = (0, 0, 0)$  and adjacent corners at  $B = (12, 0, 0)$ ,  $D = (0, 4, 0)$ , and  $E = (0, 0, 3)$ . Find coordinates for  $G$ , the corner furthest from  $A$ . Find coordinates for  $P$ , the point on segment  $AG$  that is 5 units from  $A$ .

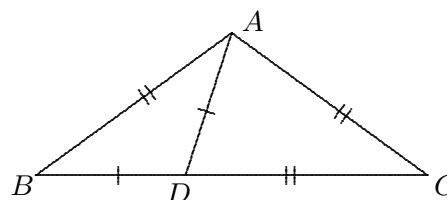
4. In the figures at right, find the sizes of the angles indicated by letters:



5. Given parallelogram  $PQRS$ , let  $T$  be the intersection of the bisectors of angles  $P$  and  $Q$ . Without knowing the sizes of the angles of  $PQRS$ , calculate the size of angle  $PTQ$ .

6. Let  $F = (3, 2)$ . There is a point  $P$  on the  $y$ -axis for which the distance from  $P$  to the  $x$ -axis equals the distance  $PF$ . Find the coordinates of  $P$ .

7. In the figure at right, it is given that  $BDC$  is straight,  $BD = DA$ , and  $AB = AC = DC$ . Find the size of angle  $C$ .



8. Mark the point  $P$  inside square  $ABCD$  that makes triangle  $CDP$  equilateral. Calculate the size of angle  $PAD$ .

9. The converse of a statement of the form “If  $\mathcal{A}$  then  $\mathcal{B}$ ” is the statement “If  $\mathcal{B}$  then  $\mathcal{A}$ .”  
 (a) Write the converse of the statement “If point  $P$  is equidistant from the coordinate axes, then point  $P$  is on the line  $y = x$ .”

(b) Give an example of a true statement whose converse is false.

(c) Give an example of a true statement whose converse is also true.

10. If a quadrilateral is a parallelogram, then both pairs of opposite angles are congruent. What is the converse of this statement? If you think that the converse is true, then prove it; if not, explain why not.

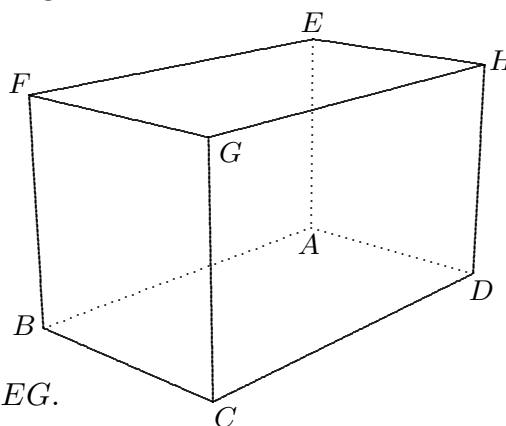
11. By making a straight cut through one vertex of an isosceles triangle, Dylan dissected the triangle into two smaller isosceles triangles. Find the angle sizes of the original triangle. There is more than one possibility. How can you be sure that you have found them all?

## Mathematics 2

1. In regular pentagon  $ABCDE$ , draw diagonal  $AC$ . What are the sizes of the angles of triangle  $ABC$ ? Prove that segments  $AC$  and  $DE$  are parallel.
2. Given square  $ABCD$ , let  $P$  and  $Q$  be the points outside the square that make triangles  $CDP$  and  $BCQ$  equilateral. Prove that triangle  $APQ$  is also equilateral.
3. The sides of an equilateral triangle are 12 cm long. How long is an altitude of this triangle? What are the angles of a right triangle created by drawing an altitude? How does the short side of this right triangle compare with the other two sides?
4. If a quadrilateral is a parallelogram, then both pairs of opposite sides are congruent. Explain. What is the converse of this statement? Is it true?
5. Although you have used the converse of the Pythagorean Theorem, it has not yet been proved in this book. State and prove the converse.
6. In triangle  $ABC$ , it is given that angle  $A$  is 59 degrees and angle  $B$  is 53 degrees. The altitude from  $B$  to line  $AC$  is extended until it intersects the line through  $A$  that is parallel to segment  $BC$ ; they meet at  $K$ . Calculate the size of angle  $AKB$ .
7. Given square  $ABCD$ , let  $P$  and  $Q$  be the points outside the square that make triangles  $CDP$  and  $BCQ$  equilateral. Segments  $AQ$  and  $BP$  intersect at  $T$ . Find angle  $ATP$ .
8. Give an example of a vector perpendicular to  $[6, 2, 3]$  that has the same length.
9. Make an accurate drawing of an acute-angled, non-equilateral triangle  $ABC$  and its circumcenter  $K$ . Use your protractor to measure (a) angles  $A$  and  $BKC$ ; (b) angles  $B$  and  $CKA$ ; (c) angles  $C$  and  $AKB$ . Do you notice anything?
10. If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram. Prove that this is so. What about the converse statement?
11. Suppose that one of the medians of a triangle happens to be exactly half the length of the side to which it is drawn. What can be said about the angles of this triangle? Justify your response.
12. (Continuation) Prove that the midpoint of the hypotenuse of a right triangle is equidistant from all three vertices of the triangle. How does this statement relate to the preceding?
13. Tate walks along the boundary of a four-sided plot of land, writing down the number of degrees turned at each corner. What is the sum of these four numbers?
14. How can one tell whether a given quadrilateral is a parallelogram? In other words, how much evidence is needed to be sure of such a conclusion?
15. In what sense is a transformation a function?

## Mathematics 2

1. Two of the corners of rectangular box  $ABCDEFGH$  are  $A = (2, 1, 3)$  and  $G = (9, 5, 7)$ . Edges  $AB$  and  $AD$  are parallel to the  $x$ - and  $y$ -axes, respectively. Add the coordinate axes to the diagram, which also illustrates questions 2 through 8.



2. Find
  - (a) coordinates for the other six vertices;
  - (b) the lengths  $AH$ ,  $AC$ ,  $AF$ ,  $FD$ , and  $AG$ ;
  - (c) the distance from  $G$  to the  $xy$ -plane;
  - (d) the distance from  $G$  to the  $z$ -axis;
  - (e) what  $C$ ,  $D$ ,  $H$ , and  $G$  have in common.
3. Is angle  $FCH$  a right angle? Explain.
4. Find the areas of quadrilaterals  $CDEF$  and  $CAEG$ .
5. Show that segments  $FD$  and  $CE$  bisect each other.
6. How many rectangles can be formed by joining four of the eight vertices?
7. A bug crawls linearly, with constant speed, from  $C$  to  $F$ , taking an hour for the trip. What are the coordinates of the bug after 24 minutes of crawling?
8. A fly flies linearly, with constant speed, from  $C$  to  $E$ , taking one minute for the flight. What are the coordinates of the fly after 24 seconds of flying?
9. Jackie walks along the boundary of a five-sided plot of land, writing down the number of degrees turned at each corner. What is the sum of these five numbers?
10. Marty walks along the boundary of a seventy-sided plot of land, writing down the number of degrees turned at each corner. What is the sum of these seventy numbers?
11. The preceding two questions illustrate the *Sentry Theorem*. What does this theorem say, and why has it been given this name?
12. A right triangle has a 24-cm perimeter, and its hypotenuse is twice as long as its shorter leg. To the nearest tenth of a cm, find the lengths of all three sides of this triangle.
13. Preparing to go on a fishing trip to Alaska, Sam wants to know whether a collapsible fishing rod will fit into a rectangular box that measures 40 inches by 20 inches by 3 inches. The longest section of the rod is 44.75 inches long. Will the rod fit in the box?
14. Alex the geologist is in the desert, 10 km from a long, straight road. Alex's jeep does 50 kph on the road and 30 kph in the desert. Alex must return immediately to base camp, which is on the same side of the road, 10 km from the road, and  $d$  km from Alex. It so happens that the quickest possible trip will take the same amount of time, whether Alex uses the road or drives all the way in the desert. Find  $d$ .

## Mathematics 2

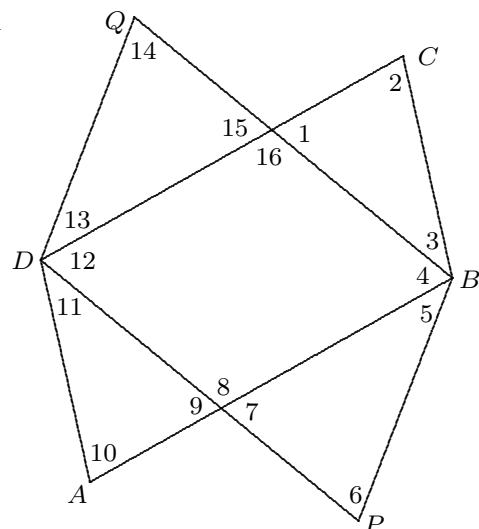
1. Let  $A = (0, 0)$ ,  $B = (8, 0)$ , and  $C = (x, x)$ . Find  $x$ , given that  
 (a)  $BC = 7$ ;      (b)  $BC = 4\sqrt{2}$ ;      (c)  $BC = 10$  and  $CAB$  is a 45-degree angle.
  
  2. Show that all the interior diagonals of a cube with 8-inch edges (a) have equal length, (b) bisect each other, and (c) are not perpendicular to each other.
- 
3. Given  $A = (7, 1, 3)$  and  $C = (4, -2, 3)$ , find the coordinates of the midpoint of segment  $AC$ .
  
  4. (Continuation) Given  $B = (5, 1, 2)$  and  $D = (6, m, n)$ , find  $m$  and  $n$  so that segments  $BD$  and  $AC$  have a common midpoint. Is  $ABCD$  a parallelogram? Explain.
  
  5. Let  $A = (1, 1)$ ,  $B = (3, 5)$ , and  $C = (7, 2)$ . Explain how to cover the whole plane with non-overlapping triangles, each of which is congruent to triangle  $ABC$ .
  
  6. (Continuation) In the pattern of lines produced by your *tesselation*, you should see triangles of many different sizes. What can you say about their sizes and shapes?
  
  7. *Midline Theorem.* Draw a triangle  $ABC$ , and let  $M$  and  $N$  be the midpoints of sides  $AB$  and  $AC$ , respectively. Express  $\overrightarrow{BC}$  and  $\overrightarrow{MN}$  in terms of  $\mathbf{u} = \overrightarrow{AB}$  and  $\mathbf{v} = \overrightarrow{AC}$ .
  
  8. Give coordinates for a point that is 8 units from the line  $y = 5$ . Then find both points on the line  $3x + 2y = 4$  that are 8 units from the line  $y = 5$ .
  
  9. Let  $F = (0, 4)$ . Find coordinates for three points that are equidistant from  $F$  and the  $x$ -axis. Write an equation that says that  $P = (x, y)$  is equidistant from  $F$  and the  $x$ -axis.
  
  10. Draw an 8-by-9-by-12 box  $ABCDEFGH$ . How many right triangles can be formed by connecting three of the eight vertices?
  
  11. Given rectangle  $ABCD$ , let  $P$  be the point outside  $ABCD$  that makes triangle  $CDP$  equilateral, and let  $Q$  be the point outside  $ABCD$  that makes triangle  $BCQ$  equilateral. Prove that triangle  $APQ$  is also equilateral.
  
  12. A regular,  $n$ -sided polygon has 18-degree exterior angles. Find the integer  $n$ .
  
  13. Let  $A = (0, 0)$ ,  $B = (7, 2)$ ,  $C = (3, 4)$ ,  $D = (3, 7)$ , and  $E = (-1, 5)$ . Cameron walks the polygonal path  $ABCDEA$ , writing down the number of degrees turned at each corner. What is the sum of these five numbers? Notice that  $ABCDE$  is *not* a *convex* pentagon.
  
  14. Is it possible for a pentagon to have interior angles  $120^\circ$ ,  $120^\circ$ ,  $120^\circ$ ,  $90^\circ$ , and  $90^\circ$ , *in this order*? What about  $120^\circ$ ,  $120^\circ$ ,  $90^\circ$ ,  $120^\circ$ , and  $90^\circ$ ? Are there other arrangements of the five angles that could have been considered? Do any of these pentagons *tesselate*?



## Mathematics 2

1. Draw a triangle  $ABC$ , and let  $AM$  and  $BN$  be two of its medians, which intersect at  $G$ . Extend  $AM$  to the point  $P$  that makes  $GM = MP$ . Prove that  $PBGC$  is a parallelogram.

2. In the figure at right, it is given that  $ABCD$  and  $PBQD$  are parallelograms. Which of the numbered angles must be the same size as the angle numbered 1?



3. Given parallelogram  $ABCD$ , with diagonals  $AC$  and  $BD$  intersecting at  $O$ , let  $POQ$  be any line with  $P$  on  $AB$  and  $Q$  on  $CD$ . Prove that  $AP = CQ$ .

4. Triangle  $PQR$  has a right angle at  $P$ . Let  $M$  be the midpoint of  $QR$ , and let  $F$  be the point where the altitude through  $P$  meets  $QR$ . Given that angle  $FPM$  is 18 degrees, find the sizes of angles  $Q$  and  $R$ .

5. Given that  $ABCDEFG \dots$  is a regular  $n$ -sided polygon, in which angle  $CAB = 12$  degrees, find  $n$ .

6. An Unidentified Flying Object (UFO) moving along a line with constant speed was sighted at  $(8, 9, 10)$  at noon and at  $(13, 19, 20)$  at 1:00 pm. Where was the UFO at 12:20 pm? When, and from where, did it leave the ground ( $z = 0$ )? What was the UFO's speed?

7. Find the image of the point  $(m, n)$  after it is reflected across the line  $y = 2x$ . After you are done, you should check your formulas. For example, they should confirm that  $(4, 3)$  and  $(0, 5)$  are images of each other. Your formulas should also confirm those points that are equal to their images — what points are these?

8. In triangle  $PQR$ , it is given that angle  $R$  measures  $r$  degrees. The bisectors of angles  $P$  and  $Q$  are drawn, creating two acute angles where they intersect. In terms of  $r$ , express the number of degrees in these acute angles.

9. Can two of the angle bisectors of a triangle intersect perpendicularly? Explain.

10. Draw triangle  $ABC$  so that angles  $A$  and  $B$  are both 42 degrees. Why should  $AB$  be longer than  $BC$ ? Extend  $CB$  to  $E$ , so that  $CB = BE$ . Mark  $D$  on  $AB$  so that  $DB = BC$ , then draw the line  $ED$ , which intersects  $AC$  at  $F$ . Find the size of angle  $CFD$ .

11. Draw a triangle  $ABC$  and two of its medians  $AM$  and  $BN$ . Let  $G$  be the point where  $AM$  intersects  $BN$ . Extend  $AM$  to the point  $P$  that makes  $GM = MP$ . Extend  $BN$  to the point  $Q$  that makes  $GN = NQ$ .

(a) Explain why  $BG$  must be parallel to  $PC$ , and  $AG$  must be parallel to  $QC$ .

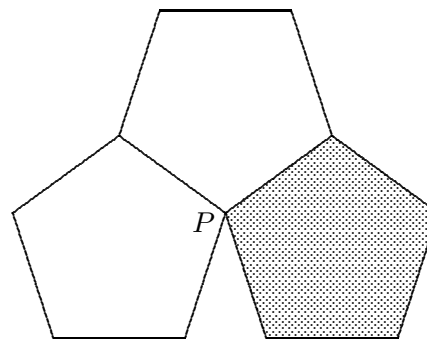
(b) What kind of a quadrilateral is  $PCQG$ ? How do you know?

(c) Find two segments in your diagram that *must* have the same length as  $BG$ .

(d) How do the lengths of segments  $BG$  and  $GN$  compare?

## Mathematics 2

1. The diagram at right shows three congruent regular pentagons that share a common vertex  $P$ . The three polygons do not quite surround  $P$ . Find the size of the uncovered acute angle at  $P$ .



2. If the shaded pentagon were removed, it could be replaced by a regular  $n$ -sided polygon that would exactly fill the remaining space. Find the value of  $n$  that makes the three polygons fit perfectly.

3. You are given a square  $ABCD$ , and midpoints  $M$  and  $N$  are marked on  $BC$  and  $CD$ , respectively. Draw  $AM$  and  $BN$ , which meet at  $Q$ . Find the size of angle  $AQB$ .

4. Mark  $Y$  inside regular pentagon  $PQRST$ , so that  $PQY$  is equilateral. Is  $RYT$  straight? Explain.

5. An airplane that took off from its airport at noon ( $t = 0$  hrs) moved according to the formula  $(x, y, z) = (15, -20, 0) + t[450, -600, 20]$ . What is the meaning of the coordinate 0 in the equation? After twelve minutes, the airplane flew over Bethlehem. Where is the airport in relation to Bethlehem, and how high (in km) was the airplane above the town?

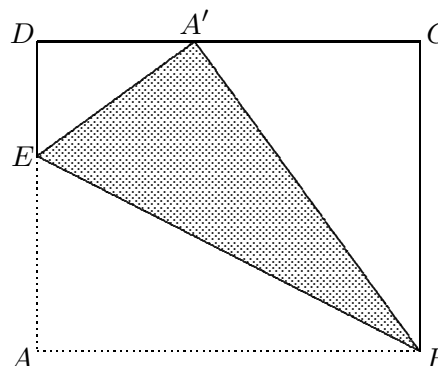
6. Suppose that triangle  $ABC$  has a right angle at  $B$ , that  $BF$  is the altitude drawn from  $B$  to  $AC$ , and that  $BN$  is the median drawn from  $B$  to  $AC$ . Find angles  $ANB$  and  $NBF$ , given that (a) angle  $C$  is 42 degrees; (b) angle  $C$  is 48 degrees.

7. Draw a parallelogram  $ABCD$ , then attach equilateral triangles  $CDP$  and  $BCQ$  to the outside of the figure. Decide whether or not triangle  $APQ$  is equilateral. Explain.

8. Suppose that  $ABCD$  is a rhombus and that the bisector of angle  $BDC$  meets side  $BC$  at  $F$ . Prove that angle  $DFC$  is three times the size of angle  $FDC$ .

9. The midpoints of the sides of a triangle are  $(3, -1)$ ,  $(4, 3)$ , and  $(0, 5)$ . Find coordinates for the vertices of the triangle.

10. In the diagram at right, a rectangular sheet of paper  $ABCD$  has been creased so that corner  $A$  is now placed on edge  $CD$ , at  $A'$ . Find the size of angle  $DEA'$ , given that the size of angle  $ABE$  is (a) 30 degrees; (b) 27 degrees; (c)  $n$  degrees.



11. Suppose that quadrilateral  $ABCD$  has the property that  $AB$  and  $CD$  are congruent and parallel. Is this enough information to prove that  $ABCD$  is a parallelogram? Explain.

## Mathematics 2

1. Suppose that square  $PQRS$  has 15-cm sides, and that  $G$  and  $H$  are on  $QR$  and  $PQ$ , respectively, so that  $PH$  and  $QG$  are both 8 cm long. Let  $T$  be the point where  $PG$  meets  $SH$ . Find the size of angle  $STG$ , with justification.
2. (Continuation) Find the lengths of  $PG$  and  $PT$ .
3. There are four special types of lines associated with triangles: Medians, perpendicular bisectors, altitudes, and angle bisectors.
  - (a) Which of these lines *must* go through the vertices of the triangle?
  - (b) Is it possible for a median to also be an altitude? Explain.
  - (c) Is it possible for an altitude to also be an angle bisector? Explain.
4. The diagonals of a rhombus have lengths 18 and 24. How long are the sides of the rhombus?
5. A *trapezoid* is a quadrilateral with exactly one pair of parallel sides. If the non-parallel sides have the same length, the trapezoid is *isosceles*. Make a diagram of an isosceles trapezoid whose sides have lengths 7 in, 10 in, 19 in, and 10 in. Find the *altitude* of this trapezoid (the distance that separates the parallel sides), then find the enclosed area.
6. Find three specific points that are equidistant from  $F = (4, 0)$  and the line  $y = x$ .
7. Draw a triangle  $ABC$  and let  $N$  be the midpoint of segment  $AC$ . Express  $\overrightarrow{BN}$  in terms of  $\mathbf{u} = \overrightarrow{AB}$  and  $\mathbf{v} = \overrightarrow{AC}$ .
8. (Continuation) Let  $M$  be the midpoint of  $BC$ . Write  $\overrightarrow{AM}$  in terms of  $\mathbf{u}$  and  $\mathbf{v}$ .
9. (Continuation) Express  $\overrightarrow{AB} + \frac{2}{3}\overrightarrow{BN}$  in terms of  $\mathbf{u}$  and  $\mathbf{v}$ . Express  $\frac{2}{3}\overrightarrow{AM}$  in terms of  $\mathbf{u}$  and  $\mathbf{v}$ . Hmm...
10. If a quadrilateral is a rectangle, then its diagonals have the same length. What is the converse of this true statement? Is the converse true? Explain.
11. The diagonals of a parallelogram always bisect each other. Is it possible for the diagonals of a trapezoid to bisect each other? Explain.
12. A trapezoid has a 60-degree angle and a 45-degree angle. What are the other angles?
13. A trapezoid has a 60-degree angle and a 120-degree angle. What are the other angles?
14. The sides of a triangle have lengths 9, 12, and 15. (This is a special triangle!)
  - (a) Find the lengths of the medians of the triangle.
  - (b) The medians intersect at the centroid of the triangle. How far is the centroid from each of the vertices of the triangle?
15. (Continuation) Apply the same questions to the equilateral triangle of side 6.

## Mathematics 2

1. Find the vertices and the area of the triangle formed by  $y = |x - 3|$  and  $-x + 2y = 5$ .
2. Trapezoid  $ABCD$  has parallel sides  $AB$  and  $CD$ , a right angle at  $D$ , and the lengths  $AB = 15$ ,  $BC = 10$ , and  $CD = 7$ . Find the length  $DA$ .
3. Equilateral triangles  $BCP$  and  $CDQ$  are attached to the outside of regular pentagon  $ABCDE$ . Is quadrilateral  $BPQD$  a parallelogram? Justify your answer.
4. A line of positive slope is drawn so that it makes a 60-degree angle where it intersects the  $x$ -axis. What is the slope of this line?
5. Mark  $P$  inside square  $ABCD$ , so that triangle  $ABP$  is equilateral. Let  $Q$  be the intersection of  $BP$  with diagonal  $AC$ . Triangle  $CPQ$  looks isosceles. Is this actually true?
6. What can be said about a quadrilateral, if it is known that every one of its adjacent-angle pairs is supplementary?
7. If  $MNPQRSTUV$  is a regular polygon, then how large is each of its interior angles? If  $MN$  and  $QP$  are extended to meet at  $A$ , then how large is angle  $PAN$ ?
8. Is it possible for the sides of a triangle to be 23, 19, and 44? Explain.
9. Suppose that  $ABCD$  is a square, with  $AB = 6$ . Let  $N$  be the midpoint of  $CD$  and  $F$  be the intersection of  $AN$  and  $BD$ . What is the length of  $AF$ ?
10. Prove that an isosceles trapezoid must have two pairs of equal adjacent angles.
11. (Continuation) The converse question: If a trapezoid has two pairs of equal adjacent angles, is it necessary that its non-parallel sides have the same length? Explain.
12. Let  $F = (2, 3)$ . Find coordinates for three points that are equidistant from  $F$  and the  $y$ -axis. Write an equation that says  $P = (x, y)$  is equidistant from  $F$  and the  $y$ -axis.
13. Write an equation for the line that is equidistant from  $5x + 3y = 15$  and  $5x + 3y = 27$ .
14. The parallel sides of trapezoid  $ABCD$  are  $AD$  and  $BC$ . Given that sides  $AB$ ,  $BC$ , and  $CD$  are each half as long as side  $AD$ , find the size of angle  $D$ .
15. Squares  $OPAL$  and  $KEPT$  are attached to the outside of equilateral triangle  $PEA$ .
  - (a) Draw segment  $TO$ , then find the size of angle  $TOP$ .
  - (b) Decide whether segments  $EO$  and  $AK$  have the same length, and give your reasons.

## Mathematics 2

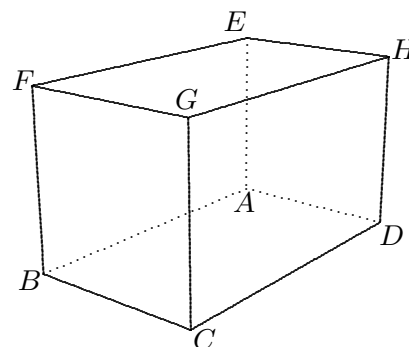
1. The lengths of the sides of triangle  $ABC$  are  $AB = 15 = AC$  and  $BC = 18$ . Find the distance from  $A$  to (a) the centroid of  $ABC$ ; (b) the circumcenter of  $ABC$ .

2. The diagram shows rectangular box  $ABCDEFGH$ , with  $A = (2, 3, 1)$ ,  $G = (10, 7, 5)$ , and edges parallel to the coordinate axes.

(a) Write parametric equations for line  $FD$ .

(b) Let  $M$  be the midpoint of segment  $CD$  and draw segment  $EM$ . Find coordinates for the point  $P$  that is two thirds of the way from  $E$  to  $M$ .

(c) Show that  $P$  is also on segment  $FD$ . Why was it predictable that segments  $FD$  and  $EM$  would intersect?



3. A parallelogram has two 19-inch sides and two 23-inch sides. What is the range of possible lengths for the diagonals of this parallelogram?

4. Is it possible for a trapezoid to have sides of lengths 3, 7, 5, and 11?

5. The altitudes of an equilateral triangle all have length 12 cm. How long are its sides?

6. It is given that the sides of an isosceles trapezoid have lengths 3 in, 15 in, 21 in, and 15 in. Make a diagram. Show that the diagonals intersect perpendicularly.

7. Triangle  $ABC$  has  $AB = AC$ . The bisector of angle  $B$  meets  $AC$  at  $D$ . Extend side  $BC$  to  $E$  so that  $CE = CD$ . Triangle  $BDE$  should look isosceles. Is it? Explain.

8. If  $ABCD$  is a quadrilateral, and  $BD$  bisects both angle  $ABC$  and angle  $CDA$ , then what sort of quadrilateral must  $ABCD$  be?

9. In quadrilateral  $ABCD$ , angles  $ABC$  and  $CDA$  are both bisected by  $BD$ , and angles  $DAB$  and  $BCD$  are both bisected by  $AC$ . What sort of quadrilateral must  $ABCD$  be?

10. Given a triangle, you have proved the following result: *The point where two medians intersect (the centroid) is twice as far from one end of a median as it is from the other end of the same median.* Improve the statement of the preceding theorem so that the reader knows which end of the median is which. This theorem indirectly shows that the three medians of any triangle must be *concurrent*. Explain the reasoning.

11. Find coordinates for the centroid of the triangle whose vertices are

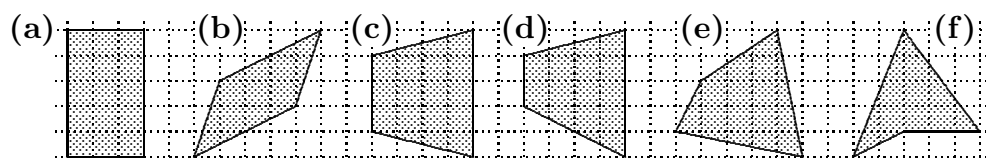
(a)  $(2, 7)$ ,  $(8, 1)$ , and  $(14, 11)$ ; (b)  $(a, p)$ ,  $(b, q)$ , and  $(c, r)$ .

12. Let  $ABCD$  be a parallelogram, with  $M$  the midpoint of  $DA$ , and diagonal  $AC$  of length 36. Let  $G$  be the intersection of  $MB$  and  $AC$ . What is the length of  $AG$ ?

13. The diagonals of a square have length 10. How long are the sides of the square?

## Mathematics 2

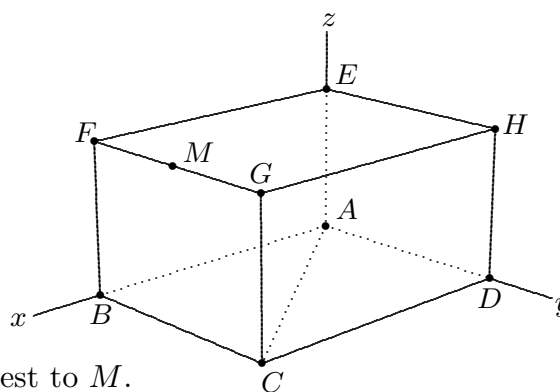
1. Triangle  $PQR$  is isosceles, with  $PQ = 13 = PR$  and  $QR = 10$ . Find the distance from  $P$  to the centroid of  $PQR$ . Find the distance from  $Q$  to the centroid of  $PQR$ .
2. (Continuation) Find the distance from  $P$  to the circumcenter of triangle  $PQR$ .
3. For what triangles is it true that the circumcenter and the centroid are the same point?
4. Is it possible for the diagonals of a parallelogram to have the same length? How about the diagonals of a trapezoid? How about the diagonals of a non-isosceles trapezoid?
5. Calculate the effect of the transformation  $\mathcal{T}(x, y) = \left(\frac{3}{5}x + \frac{4}{5}y, \frac{4}{5}x - \frac{3}{5}y\right)$  on a triangle of your choosing. Is  $\mathcal{T}$  an isometry? If so, what kind?
6. How many diagonals can be drawn inside a pentagon? a hexagon? a decagon? a twenty-sided polygon? an  $n$ -sided polygon?
7. Suppose that  $P$  is twice as far from  $(0, 0)$  as  $P$  is from  $(6, 0)$ . Find such a point on the  $x$ -axis. Find another such point that is *not* on the  $x$ -axis.
8. Let  $A = (1.43, 10.91)$ ,  $B = (3.77, 7.33)$ , and  $C = (8.15, 2.55)$ . Find coordinates for  $G$ , the centroid of triangle  $ABC$ . Find an equation for the line through  $G$  parallel to  $AC$ .
9. Let  $A = (0, 0)$ ,  $B = (6, 2)$ ,  $C = (-5, 2)$ ,  $D = (7, 6)$ ,  $E = (-1, 7)$ , and  $F = (5, 9)$ . Draw lines  $AB$ ,  $CD$ , and  $EF$ . Verify that they are parallel.
  - (a) Draw the transversal of slope  $-1$  that goes through  $E$ . This transversal intersects line  $AB$  at  $G$  and line  $CD$  at  $H$ . Use your ruler to measure  $EH$  and  $HG$ .
  - (b) Draw any transversal whose slope is  $2$  (it need not go through  $E$ ), intersecting line  $AB$  at  $P$ , line  $CD$  at  $Q$ , and line  $EF$  at  $R$ . Use your ruler to measure  $PQ$  and  $QR$ .
  - (c) On the basis of your findings, propose a theorem. A proof is not requested.
10. In triangle  $ABC$ , let  $M$  be the midpoint of  $AB$  and  $N$  be the midpoint of  $AC$ . Suppose that you measure  $MN$  and find it to be  $7.3$  cm long. How long would  $BC$  be, if you measured it? If you were to measure angles  $AMN$  and  $ABC$ , what would you find?
11. In triangle  $TOM$ , let  $P$  and  $Q$  be the midpoints of segments  $TO$  and  $TM$ , respectively. Draw the line through  $P$  parallel to segment  $TM$ , and the line through  $Q$  parallel to segment  $TO$ ; these lines intersect at  $J$ . What can you say about the location of point  $J$ ?
12. Let  $G$  be the centroid of triangle  $ABC$ . Simplify the vector sum  $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}$ .
13. Which of these quadrilaterals can be used to tessellate a plane? Justify your choices.



## Mathematics 2

1. Mark  $A = (0, 0)$  and  $B = (10, 0)$  on your graph paper, and use your protractor to draw the line of positive slope through  $A$  that makes a 25-degree angle with  $AB$ . By making suitable measurements, calculate (approximately) the slope of this line.
2. (Continuation) Turn on your calculator, press the MODE button, and select the *Degree* option for angles. Return to the home screen, and ENTER the expression  $\text{TAN}(25)$ . You should see that the display agrees with your answer to the preceding item.
3. Repeat the preceding construction and TAN verification for at least three more angles of your choosing.
4. Find coordinates for a point that is 5 units from the line  $3x + 4y = 10$ .

5. The diagram at right shows rectangular box  $ABCDEFGH$ , with  $A = (0, 0, 0)$ ,  $G = (4, 3, 2)$ , and the sides parallel to the coordinate axes. The midpoint of  $FG$  is  $M$ .



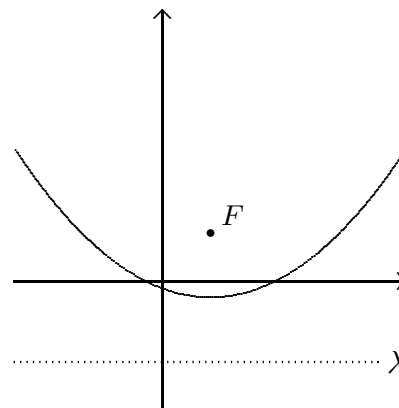
- (a) Find coordinates for  $M$ .
- (b) Find coordinates for the point  $P$  on segment  $AC$  that is 2 units from  $A$ .
- (c) Decide whether angle  $APM$  is a right angle, and give your reasons.
- (d) Find the point on segment  $AC$  that is closest to  $M$ .
6. Find coordinates for the centroid of the triangle whose vertices are  $(2, 1, 3)$ ,  $(4, 5, 6)$ , and  $(0, 3, 1)$ .
7. Let  $A = (0, 0)$ ,  $B = (12, 4)$ ,  $C = (2, 3)$ ,  $D = (8, 5)$ ,  $E = (5, -3)$ , and  $F = (11, -1)$ . Draw lines  $AB$ ,  $CD$ , and  $EF$ . Verify that they are parallel.
  - (a) Draw the transversal of slope  $-1$  that goes through  $F$ . This transversal intersects line  $AB$  at  $G$  and line  $CD$  at  $H$ . Use your ruler to measure  $FG$  and  $GH$ .
  - (b) Draw any transversal whose slope is 3, and let  $P$ ,  $Q$ , and  $R$  be its intersections with lines  $AB$ ,  $CD$ , and  $EF$ , respectively. Use your ruler to measure  $PQ$  and  $PR$ .
  - (c) On the basis of your findings, formulate the *Three Parallels Theorem*.
8. A line drawn parallel to the side  $BC$  of triangle  $ABC$  intersects side  $AB$  at  $P$  and side  $AC$  at  $Q$ . The measurements  $AP = 3.8$  in,  $PB = 7.6$  in, and  $AQ = 5.6$  in are made. If segment  $QC$  were now measured, how long would it be?
9. Draw an acute-angled triangle  $ABC$ , and mark points  $P$  and  $Q$  on sides  $AB$  and  $AC$ , respectively, so that  $AB = 3AP$  and  $AC = 3AQ$ . Express  $\overrightarrow{PQ}$  and  $\overrightarrow{BC}$  in terms of  $\mathbf{v} = \overrightarrow{AP}$  and  $\mathbf{w} = \overrightarrow{AQ}$ .
10. Let  $A = (7, 1, 1)$  and  $B = (-3, 2, 7)$ . Find all the points  $P$  on the  $z$ -axis that make angle  $APB$  right.

## Mathematics 2

1. Let  $F = (3, 3)$ . The points  $P = (x, y)$  that are equidistant from  $F$  and the line  $y = -5$  form a parabola. Find the points where this parabola meets the  $x$ -axis.

2. (Continuation) Find an equation for the parabola.

3. Given a line  $\lambda$  (Greek “lambda”) and a point  $F$  not on  $\lambda$ , let  $P$  be on the parabola of all points that are equidistant from  $F$  and from  $\lambda$ . Let  $N$  be the point on  $\lambda$  closest to  $P$ . Prove that the parabola lies on one side of the perpendicular bisector of  $FN$ .



4. Is it possible for a scalene triangle to have two medians of the same length? Explain.

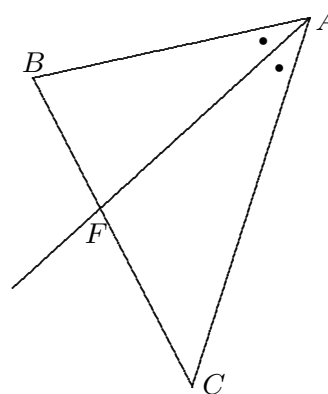
5. Standing 50 meters from the base of a fir tree, Rory used a protractor to measure an *angle of elevation* of  $33^\circ$  to the top of the tree. How tall was the tree?

6. Given  $A = (0, 6)$ ,  $B = (-8, 0)$ , and  $C = (8, 0)$ , find coordinates for the circumcenter of triangle  $ABC$ .

7. Given regular hexagon  $BAGELS$ , show that  $SEA$  is an equilateral triangle.

8. Let  $A = (0, 0)$ ,  $B = (0, 3)$ , and  $C = (4, 0)$ . Let  $F$  be the point where the bisector of angle  $BAC$  meets side  $BC$ . Find exact coordinates for  $F$ . Notice that  $F$  is not the midpoint of  $BC$ . Finally, calculate the distances  $BF$  and  $CF$ . Do you notice anything?

9. (Continuation) Draw an acute-angled, scalene, lattice triangle  $ABC$  of your choosing, then use your protractor to carefully draw the bisector of angle  $BAC$ . Let  $F$  be the intersection of the bisector with  $BC$ . Measure the lengths  $AB$ ,  $AC$ ,  $FB$ , and  $FC$ . Do you notice anything?



10. Choose four lattice points for the vertices of a non-isosceles trapezoid  $ABCD$ , with  $AB$  longer than  $CD$  and parallel to  $CD$ . Extend  $AD$  and  $BC$  until they meet at  $E$ . Verify that the ratios  $\frac{AD}{DE}$  and  $\frac{BC}{CE}$  are equal, by measurement or calculation.

11. Segments  $AC$  and  $BD$  intersect at  $E$ , so as to make  $AE$  twice  $EC$  and  $BE$  twice  $ED$ . Prove that segment  $AB$  is twice as long as segment  $CD$ , and parallel to it.

12. A rectangular box is 2 by 3 by  $h$ , and two of its internal diagonals are perpendicular. Find possible values for  $h$ .

13. When the Sun has risen 32 degrees above the horizon, a Lower casts a shadow that is 9 feet 2 inches long. How tall is the Lower, to the nearest inch?



## Mathematics 2

0. In the following list of true statements, find (a) the statements whose converses are also in the list; (b) the statement that is a definition; (c) a statement whose converse is false; (d) the Sentry Theorem; and (e) the Midline Theorem.

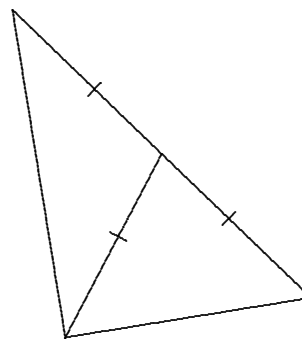
1. If a quadrilateral has two pairs of parallel sides, then its diagonals bisect each other.
2. If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral must be a parallelogram.

3. If a quadrilateral is equilateral, then it is a rhombus.

4. If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

5. If a quadrilateral has two pairs of equal adjacent sides, then its diagonals are perpendicular.

6. If one of the medians of a triangle is half the length of the side to which it is drawn, then the triangle is a right triangle.



7. If a segment joins two of the midpoints of the sides of a triangle, then it is parallel to the third side, and is half the length of the third side.

8. Both pairs of opposite sides of a parallelogram are congruent.

9. The sum of the exterior angles of any polygon — one at each vertex — is 360 degrees.

10. The median drawn to the hypotenuse of a right triangle is half the length of the hypotenuse.

11. If two lines are intersected by a transversal so that alternate interior angles are equal, then the lines must be parallel.

12. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is in fact a parallelogram.

13. If two opposite sides of a quadrilateral are both parallel and equal in length, then the quadrilateral is a parallelogram.

14. If three parallel lines intercept equal segments on one transversal, then they intercept equal segments on every transversal.

15. Both pairs of opposite angles of a parallelogram are congruent.

16. The medians of any triangle are concurrent, at a point that is two thirds of the way from any vertex to the midpoint of the opposite side.

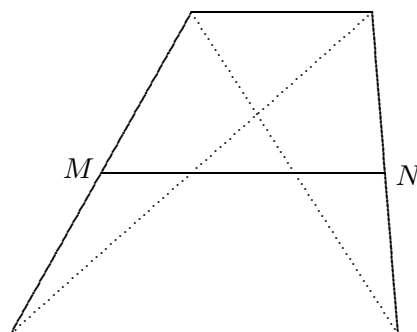
17. An exterior angle of a triangle is the sum of the two nonadjacent interior angles.

## Mathematics 2

1. The position of an airplane that is approaching its airport is described parametrically by  $P_t = (1000, 500, 900) + t[-100, -50, -90]$ . For what value of  $t$  is the airplane closest to the traffic control center located at  $(34, 68, 16)$ ?
2. Triangle  $ABC$  has  $AB = 10 = AC$  and  $BC = 12$ . Find the distance from  $A$  to
  - (a) the centroid of  $ABC$ ;
  - (b) the circumcenter of  $ABC$ .
3. Suppose that a quadrilateral is measured and found to have a pair of equal nonadjacent sides and a pair of equal nonadjacent angles. Is this enough evidence to conclude that the quadrilateral is a parallelogram? Explain.
4. Suppose that  $ANGEL$  is a regular pentagon, and that  $CANT$ ,  $HALF$ ,  $ROLE$ ,  $KEGS$ , and  $PING$  are squares attached to the outside of the pentagon. Show that decagon  $PITCHFORKS$  is equiangular. Is this decagon equilateral?
5. Let  $P = (-15, 0)$ ,  $Q = (5, 0)$ ,  $R = (8, 21)$ , and  $S = (0, 15)$ . Draw quadrilateral  $PQRS$  and measure its sides and angles. Is there anything remarkable about this figure?
6. Rearrange the letters of *doctrine* to spell a familiar mathematical word.
7. Given triangle  $ABC$ , let  $F$  be the point where segment  $BC$  meets the bisector of angle  $BAC$ . Draw the line through  $B$  that is parallel to segment  $AF$ , and let  $E$  be the point where this parallel meets the extension of segment  $CA$ .
  - (a) Find the four congruent angles in your diagram.
  - (b) How are the lengths  $EA$ ,  $AC$ ,  $BF$ , and  $FC$  related?
  - (c) *The Angle-Bisector Theorem*: How are the lengths  $AB$ ,  $AC$ ,  $BF$ , and  $FC$  related?
8. Standing on a cliff 380 meters above the sea, Pat sees an approaching ship and measures its *angle of depression*, obtaining 9 degrees. How far from shore is the ship?
9. (Continuation) Now Pat sights a second ship beyond the first. The angle of depression of the second ship is 5 degrees. How far apart are the ships?
10. Let  $RICK$  be a parallelogram, with  $M$  the midpoint of  $RI$ . Draw the line through  $R$  that is parallel to  $MC$ ; it meets the extension of  $IC$  at  $P$ . Prove that  $CP = KR$ .
11. Suppose that  $ABCD$  is a trapezoid, with  $AB$  parallel to  $CD$ . Let  $M$  and  $N$  be the midpoints of  $DA$  and  $BC$ , respectively. What can be said about segment  $MN$ ? Explain.
12. What is the radius of the smallest circle that encloses an equilateral triangle with 12-inch sides? What is the radius of the largest circle that will fit inside the same triangle?
13. Suppose that  $ASCENT$  is a regular hexagon, and that  $ARMS$ ,  $BATH$ ,  $LINT$ ,  $FEND$ ,  $COVE$ , and  $CUPS$  are squares attached to the outside of the hexagon. Decide whether or not dodecagon  $LIDFVOUPMRBH$  is regular, and give your reasons.

## Mathematics 2

1. In acute triangle  $ABC$ , the bisector of angle  $ABC$  meets side  $AC$  at  $D$ . Mark points  $P$  and  $Q$  on sides  $BA$  and  $BC$ , respectively, so that segment  $DP$  is perpendicular to  $BA$  and segment  $DQ$  is perpendicular to  $BC$ . Prove that triangles  $BDP$  and  $BDQ$  are congruent. What about triangles  $PAD$  and  $QCD$ ?
2. Let  $A = (0, 0)$ ,  $B = (4, 0)$ , and  $C = (4, 3)$ . Measure angle  $CAB$ . What is the slope of  $AC$ ? Use this slope and the TAN button to check your angle measurement. Use your calculator to come as close as you can to the theoretically correct size of angle  $CAB$ .
3. (Continuation) On your calculator, ENTER the expression  $\text{TAN}^{-1}(0.75)$ . Hmm...
4. Find coordinates for the point  $P$  where the line  $y = x$  intersects the line  $2x + 3y = 24$ . Then calculate the distances from  $P$  to the axis intercepts of  $2x + 3y = 24$ . The Angle-Bisector Theorem makes a prediction about these distances — what is the prediction?
5. A five-foot Upper casts an eight-foot shadow. How high is the Sun in the sky? This question is not asking for a distance, by the way.
6. Are the points  $(2, 5, 7)$ ,  $(12, 25, 37)$ , and  $(27, 55, 81)$  collinear?
7. Find coordinates for the point where the line  $(x, y, z) = (7 + 2r, 5 - 3r, 4 + r)$  intersects the  $xz$ -plane.
8. Inside regular pentagon  $JERZY$  is marked point  $P$  so that triangle  $JEP$  is equilateral. Decide whether or not quadrilateral  $JERP$  is a parallelogram, and give your reasons.
9. Suppose that  $ABCD$  is a parallelogram, in which  $AB = 2BC$ . Let  $M$  be the midpoint of segment  $AB$ . Prove that segments  $CM$  and  $DM$  bisect angles  $BCD$  and  $CDA$ , respectively. What is the size of angle  $CMD$ ? Justify your response.
10. If  $M$  and  $N$  are the midpoints of the non-parallel sides of a trapezoid, it makes sense to call the segment  $MN$  the *midline* of the trapezoid. Why? (It actually should be called the *midsegment*, of course. Strange to say, some textbooks call it the *median*). Suppose that the parallel sides of a trapezoid have lengths 7 and 15. What is the length of the midline of the trapezoid? What are the lengths of the three pieces into which the midline is divided by the points where it intersects the diagonals of the trapezoid?



11. An isosceles trapezoid has sides of lengths 9, 10, 21, and 10. Find the distance that separates the parallel sides, then find the length of the diagonals. Finally, find the angles of the trapezoid, to the nearest tenth of a degree.
12. Find the angle formed by the diagonal of a cube and a diagonal of a face of the cube.

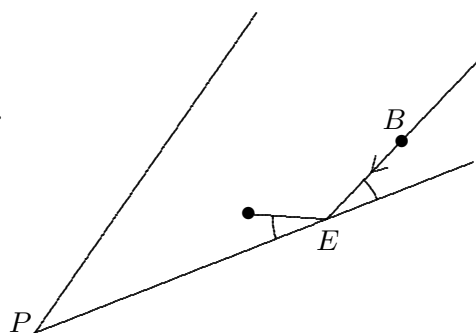
## Mathematics 2

1. One day at the beach, Kelly flies a kite, whose string makes a 37-degree elevation angle with the ground. Kelly is 130 feet from the point directly below the kite. How high above the ground is the kite, to the nearest foot?
2. Think of the ground you are standing on as the  $xy$ -plane. The vector  $[12, 5, 10]$  points from you toward the Sun. How high is the Sun in the sky?
3. Hexagon  $ABCDEF$  is regular. Prove that segments  $AE$  and  $ED$  are perpendicular.
4. Suppose that  $PQRS$  is a rhombus, with  $PQ = 12$  and a 60-degree angle at  $Q$ . How long are the diagonals  $PR$  and  $QS$ ?
5. A triangle, whose sides are 6, 8, and 10, and a circle, whose radius is  $r$ , are drawn so that no part of the triangle lies outside the circle. How small can  $r$  be?
6. Let  $ABCD$  be a square. Mark midpoints  $M$ ,  $N$ ,  $O$ , and  $P$  on  $AB$ ,  $BC$ ,  $CD$ , and  $DA$ , respectively. Draw  $AN$ ,  $BO$ ,  $CP$ , and  $DM$ . Let  $Q$  and  $R$  be the intersections of  $AN$  with  $DM$  and  $BO$ , respectively, and let  $S$  and  $T$  be the intersections of  $CP$  with  $BO$  and  $DM$ , respectively. Prove as much as you can about this figure, especially quadrilateral  $QRST$ .
7. (Continuation) Segment  $AB$  is 10 cm long. How long is  $QR$ , to the nearest 0.1 cm?
8. Diagonals  $AC$  and  $BD$  of regular pentagon  $ABCDE$  intersect at  $H$ . Decide whether or not  $AHDE$  is a rhombus, and give your reasons.
9. Do the lines  $(x, y, z) = (5 + 2t, 3 + 2t, 1 - t)$  and  $(x, y, z) = (13 - 3r, 13 - 4r, 4 - 2r)$  intersect? If so, at what point? If not, how do you know?
10. *The Doppler Shift.* While driving a car, AJ honks the horn every 5 seconds. Hitch is standing by the side of the road, and hears the honks of the oncoming car every 4.6 seconds. The speed of sound is 330 meters per second. Calculate the speed of AJ's car. Describe what Hitch hears after the car passes.
11. Let  $A = (3, 1)$ ,  $B = (9, 5)$ , and  $C = (4, 6)$ . Your protractor should tell you that angle  $CAB$  is about 45 degrees. Explain why angle  $CAB$  is in fact exactly 45 degrees.
12. Draw a regular pentagon and all five of its diagonals. How many isosceles triangles can you find in your picture? How many scalene triangles can you find?
13. The sides of a triangle are 6 cm, 8 cm, and 10 cm long. Find the distances from the centroid of this triangle to the three vertices.
14. The diagonals of a non-isosceles trapezoid divide the midline into three segments, whose lengths are 8 cm, 3 cm, and 8 cm. How long are the parallel sides? From this information, is it possible to infer anything about the distance that separates the parallel sides? Explain.

## Mathematics 2

1. The sides of a polygon are cyclically extended to form *rays*, creating one exterior angle at each vertex. Viewed from a great distance, what theorem does this figure illustrate?

2. The diagram at right shows one corner of a triangular billiards table. A ball leaves point  $B$  and follows the indicated path, striking the edge of the table at  $E$ . Thereafter, at each impact, the ball obeys the *law of reflection*, which says that the incoming angle equals the outgoing angle. Given that there is a 34-degree angle at corner  $P$ , and that the initial impact makes a 25-degree angle at  $E$ , how many bounces will the ball make before its path leaves this page?



3. Given a rectangular card that is 5 inches long and 3 inches wide, what does it mean for another rectangular card to have the *same shape*? Describe a couple of examples.

4. A rectangle is 2 inches wide, and more than 2 inches long. It so happens that this rectangle can be divided, by a single cut, into a 2-inch square and a small rectangle that has exactly the same shape as the large rectangle. What is the length of the large rectangle?

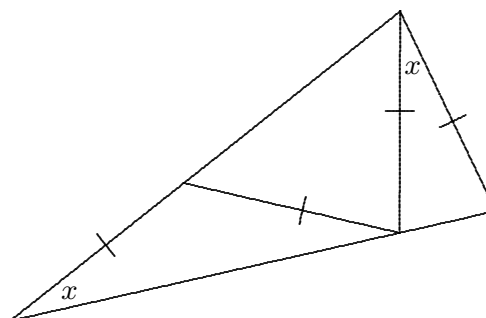
5. Let  $A = (1, 2, 3)$ ,  $B = (3, 7, 9)$ , and  $D = (-2, 3, -1)$ . Find coordinates for vertex  $C$  of parallelogram  $ABCD$ . How many parallelograms can you find that have the three given vertices among their four vertices?

6. *The Orthocenter.* Given an acute-angled triangle  $ABC$ , draw the line through  $A$  parallel to  $BC$ , the line through  $B$  parallel to  $AC$ , and the line through  $C$  parallel to  $AB$ . These lines form triangle  $PQR$ . The altitudes of triangle  $ABC$  are also special lines for triangle  $PQR$ . Explain.

7. In trapezoid  $ABCD$ ,  $AB$  is parallel to  $CD$ , and  $AB = 10$ ,  $BC = 9$ ,  $CD = 22$ , and  $DA = 15$ . Points  $P$  and  $Q$  are marked on  $BC$  so that  $BP = PQ = QC = 3$ , and points  $R$  and  $S$  are marked on  $DA$  so that  $DR = RS = SA = 5$ . Find the lengths  $PS$  and  $QR$ .

8. *The Varignon quadrilateral.* A quadrilateral has diagonals of lengths 8 and 10. The midpoints of the sides of this figure are joined to form a new quadrilateral. What is the perimeter of the new quadrilateral? What is special about it?

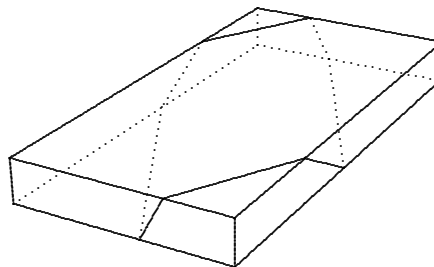
9. In the figure at right, there are two  $x$ -degree angles, and four of the segments are congruent as marked. Find  $x$ .



10. The parallel bases of a trapezoid have lengths 12 and 18 cm. Find the lengths of the two segments into which the midline of the trapezoid is divided by a diagonal.

## Mathematics 2

1. In triangle  $ABC$ , points  $M$  and  $N$  are marked on sides  $AB$  and  $AC$ , respectively, so that  $AM:AB = 17:100 = AN:AC$ . Show that segments  $MN$  and  $BC$  are parallel.
2. (Continuation) In triangle  $ABC$ , points  $M$  and  $N$  are marked on sides  $AB$  and  $AC$ , respectively, so that the ratios  $AM:AB$  and  $AN:AC$  are both equal to  $r$ , where  $r$  is some number between 0 and 1. Show that segments  $MN$  and  $BC$  are parallel.
3. A cheetah can run at 105 feet per second, but only for 7 seconds, at which time the animal must stop and rest. A fully rested cheetah at  $(0, 0)$  notices a nearby antelope, which is moving according to the parametric equation  $(x, y) = (-39 + 40t, 228 + 30t)$ , where  $t$  is measured in seconds and  $x$  and  $y$  are measured in feet. If it started to run at  $t = 0$ , the cheetah could catch the antelope. For how many more seconds can the cheetah afford to wait before starting? Assume that the cheetah does not change direction when it runs.
4. The diagonals of rhombus  $ABCD$  meet at  $M$ . Angle  $DAB$  measures 60 degrees. Let  $P$  be the midpoint of  $AD$ , and let  $G$  be the intersection of  $PC$  and  $MD$ . Given that  $AP = 8$ , find  $MD$ ,  $MC$ ,  $MG$ ,  $CG$ , and  $GP$ .
5. Rectangle  $ABCD$  has dimensions  $AB = 5$  and  $BC = 12$ . Let  $M$  be the midpoint of  $BC$ , and let  $G$  be the intersection of  $AM$  and diagonal  $BD$ . Find  $BG$  and  $AG$ .
6. Show that a regular dodecagon can be cut into pieces that are all regular polygons, which need not all have the same number of sides.
7. The hypotenuse of a right triangle is twice as long as one of the legs. How long is the other leg? What is the size of the smallest angle?
8. What is the smallest amount of ribbon that is needed to wrap around a  $2'' \times 10'' \times 20''$  gift box in the way shown in the figure at right? You could experiment with some string and a book.
9. What are the angle sizes in a trapezoid whose sides have lengths 6, 20, 6, and 26?
10. Given square  $ABCD$ , choose a point  $O$  that is not outside the square and form the vector  $\mathbf{v} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$ . By trying various possible positions for  $O$ , find the shortest and longest possible  $\mathbf{v}$ .
11. Alden, a passenger on a yacht moored 15 miles due north of a straight, east-west coastline, has become ill and has to be taken ashore in a small motorboat, which will meet an ambulance at some point on the shore. The ambulance will then take Alden to the hospital, which is 60 miles east of the shore point closest to the yacht. The motorboat can travel at 20 mph and the ambulance at 90 mph. In what direction should the motorboat head, to minimize the travel time to the hospital? Express your answer using an angle.



## Mathematics 2

1. Find the angle formed by two face diagonals that intersect at a vertex of the cube.
2. Let  $A = (0, 0)$ ,  $B = (4, -3)$ ,  $C = (6, 3)$ ,  $P = (-2, 7)$ ,  $Q = (9, 5)$ , and  $R = (7, 19)$ . Measure the angles of triangles  $ABC$  and  $PQR$ . Calculate the lengths of the sides of these triangles. Find justification for any conclusions you make.
3. Find components for the vector that points from  $A$  to  $B$  when
  - (a)  $A = (2, 5, -6)$  and  $B = (x, y, z)$ ;
  - (b)  $A = (4, 1 + 2t, 3 - t)$  and  $B = (1, 5, 0)$ .
4. Find coordinates for a point that is 7 units from the line  $4x + y = 11$ .
5. While the Wood's Hole-Martha's Vineyard ferryboat steamed along at 8 mph through calm seas, passenger Dale exercised by walking the perimeter of the rectangular deck, at a steady 4 mph. Discuss the variations in Dale's speed *relative to the water*.
6. Brett and Jordan are cruising, according to the equations  $B_t = (27 + 4t, 68 - 7t, 70 + 4t)$  and  $J_t = (23 + 4t, 11 + t, 34 + 8t)$ . Show that their paths intersect, but that there is no collision. Who reaches the intersection first? Who is moving faster?
7. Let  $A = (4, 0, 0)$ ,  $B = (0, 3, 0)$ , and  $C = (0, 0, 5)$ .
  - (a) Draw a diagram, then find coordinates for the point closest to  $C$  on segment  $AB$ .
  - (b) Find the area of triangle  $ABC$ .
  - (c) Find the length of the altitude drawn from  $A$  to  $BC$ .
8. In triangle  $ABC$ , it is given that  $AB = 4$ ,  $AC = 6$ , and  $BC = 5$ . The bisector of angle  $BAC$  meets  $BC$  at  $D$ . Find lengths  $BD$  and  $CD$ .
9. Let  $A = (1, 2)$ ,  $B = (8, 2)$ , and  $C = (7, 10)$ . Find an equation for the line that bisects angle  $BAC$ .
10. Atiba wants to measure the width of the Squamscott River. Standing under a tree  $T$  on the river bank, Atiba sights a rock at the nearest point  $R$  on the opposite bank. Then Atiba walks to a point  $P$  on the river bank that is 50.0 meters from  $T$ , and makes  $RTP$  a right angle. Atiba then measures  $RPT$  and obtains 76.8 degrees. How wide is the river?
11. Let  $P$  be the circumcenter and  $G$  be the centroid of a triangle formed by placing two perpendicular vectors  $\mathbf{v}$  and  $\mathbf{w}$  tail to tail. Express  $\overrightarrow{GP}$  in terms of vectors  $\mathbf{v}$  and  $\mathbf{w}$ .
12. A regular  $n$ -sided polygon has exterior angles of  $m$  degrees each. Express  $m$  in terms of  $n$ . For how many of these regular examples is  $m$  a whole number?
13. Out for a walk in Chicago, Morgan measures the angle of elevation to the distant Willis Tower, and gets 3.6 degrees. After walking one km directly toward the building, Morgan finds that the angle of elevation has increased to 4.2 degrees. Use this information to calculate the height of the Willis Tower, and how far Morgan is from it now.

## Mathematics 2

1. Alex the geologist is in the desert, 18 km from a long, straight road and 72 km from base camp, which is also 18 km from the road, on the same side of the road as Alex is. On the road, the jeep can do 60 kph, but in the desert sands, it can only manage 32 kph.

(a) Describe the path that Alex should follow, to return to base camp most quickly.

(b) If the jeep were capable of 40 kph in the desert, how would your answer be affected?

2. Triangle  $ABC$  has  $AB = 12 = AC$  and angle  $A$  is 120 degrees. Let  $F$  and  $D$  be the midpoints of sides  $AC$  and  $BC$ , respectively, and  $G$  be the intersection of segments  $AD$  and  $BF$ . Find the lengths  $FD$ ,  $AD$ ,  $AG$ ,  $BG$ , and  $BF$ .

3. Simplify the vector expression  $\overrightarrow{AB} - \overrightarrow{AC}$ , and illustrate with a diagram.

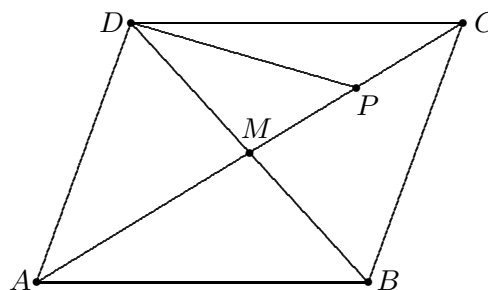
4. Find the side of the largest square that can be drawn inside a 12-inch equilateral triangle, one side of the square aligned with one side of the triangle.

5. How tall is an isosceles triangle, given that its base is 30 cm long and that both of its base angles are 72 degrees?

6. To the nearest tenth of a degree, how large are the congruent angles of an isosceles triangle that is exactly as tall as it is wide? (There is more than one interpretation.)

7. In the figure at right,  $ABCD$  is a parallelogram, with diagonals  $AC$  and  $BD$  intersecting at  $M$ , and  $P$  the midpoint of  $CM$ . Express the following in terms of  $\mathbf{u} = \overrightarrow{AB}$  and  $\mathbf{v} = \overrightarrow{AD}$ :

(a)  $\overrightarrow{AM}$  (b)  $\overrightarrow{BD}$  (c)  $\overrightarrow{CP}$  (d)  $\overrightarrow{DP}$



8. Rectangle  $ABCD$  has  $AB = 16$  and  $BC = 6$ . Let  $M$  be the midpoint of side  $AD$  and  $N$  be the midpoint of side  $CD$ . Segments  $CM$  and  $AN$  intersect at  $G$ . Find the length  $AG$ .

9. The parallel sides of a trapezoid have lengths  $m$  and  $n$  cm. In terms of  $m$  and  $n$ , how long are the three pieces into which the midline of the trapezoid is divided by the diagonals?

10. To the nearest tenth of a degree, find the sizes of the acute angles in the 5-12-13 triangle and in the 9-12-15 triangle. This enables you to calculate the sizes of the angles in the 13-14-15 triangle. Show how to do it, then invent another example of this sort.

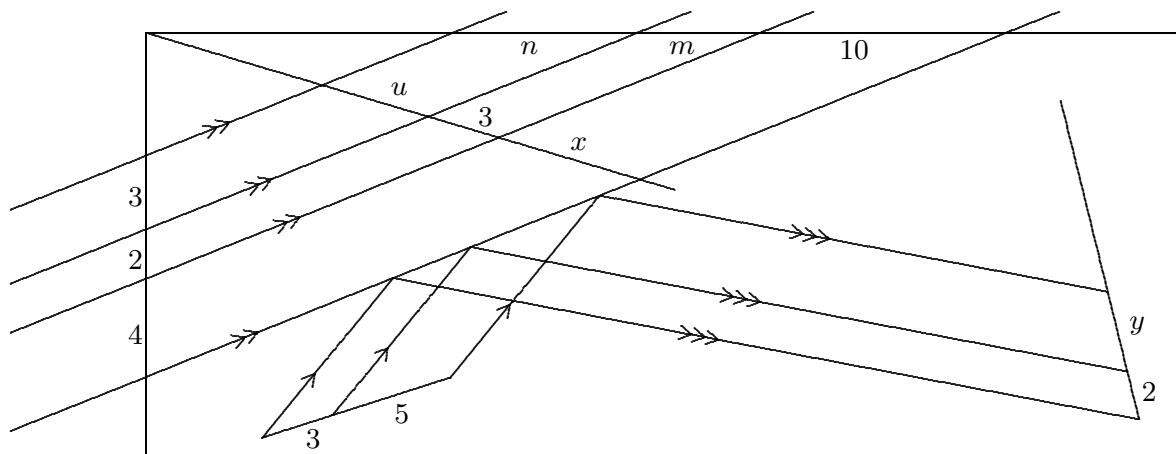
11. A triangle has a 60-degree angle and a 45-degree angle, and the side opposite the 45-degree angle is 240 mm long. To the nearest mm, how long is the side opposite the 60-degree angle?

12. Explain how two congruent trapezoids can be combined without overlapping to form a parallelogram. What does this tell you about the length of the midline of the trapezoid?

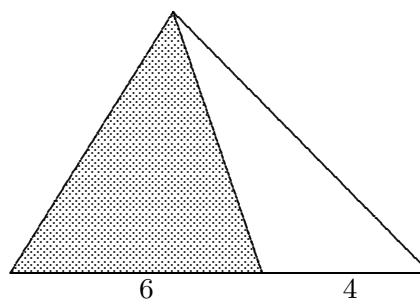


## Mathematics 2

1. In both of the figures below, find the lengths of the segments indicated by letters. Parallel lines are indicated by arrows.



2. In 1983, there were 975 students at P.E.A, and the girl:boy ratio was 2:3.
- How many students were girls?
  - How many boys would you expect to find in a class of fifteen students? Explain.
3. An estate of \$362880 is to be divided among three heirs, Alden, Blair, and Cary. According to the will, Alden is to get two parts, Blair three parts, and Cary four parts. What does this mean, in dollars and cents?
4. Given that  $P$  is three fifths of the way from  $A$  to  $B$ , and that  $Q$  is one third of the way from  $P$  to  $B$ , describe the location of  $Q$  in relation to  $A$  and  $B$ .
5. Suppose that the points  $A$ ,  $P$ ,  $Q$ , and  $B$  appear in this order on a line, such that  $AP:AB = 3:5$  and  $PQ:QB = 1:2$ . Evaluate the ratios  $AQ:AB$  and  $AQ:QB$ .
6. Apply the transformation  $\mathcal{T}(x, y) = (3x, 3y)$  to the triangle  $PQR$  whose vertices are  $P = (3, -1)$ ,  $Q = (1, 2)$ , and  $R = (4, 3)$ . Compare the sides and angles of the image triangle  $P'Q'R'$  with the corresponding parts of  $PQR$ . This transformation is an example of a *dilation*, also called a *radial expansion*. Is  $\mathcal{T}$  an isometry?
7. In the figure at right, the shaded triangle has area 15. Find the area of the unshaded triangle.
8. In a triangle whose sides have lengths 3, 4, and 5,
- how long is the bisector of the larger of the two acute angles?
  - how long is the bisector of the right angle?
9. Show that the altitude drawn to the hypotenuse of any right triangle divides the triangle into two triangles that have the same angles as the original.



## Mathematics 2

1. To the nearest tenth of a degree, find the sizes of the acute angles in the 7-24-25 right triangle and in the 8-15-17 right triangle. This information then allows you to calculate the sizes of all the angles in the 25-51-52 triangle. Show how to do it.

2. Compare the quadrilateral whose vertices are  $A = (0, 0)$ ,  $B = (6, 2)$ ,  $C = (5, 5)$ ,  $D = (-1, 3)$  with the quadrilateral whose vertices are  $P = (9, 0)$ ,  $Q = (9, 2)$ ,  $R = (8, 2)$ , and  $S = (8, 0)$ . Calculate lengths and angles, and look for patterns.

3. Draw a right triangle that has a 15-cm hypotenuse and a 27-degree angle. To the nearest tenth of a cm, measure the side opposite the 27-degree angle, and then express your answer as a percentage of the length of the hypotenuse. Compare your answer with the value obtained from your calculator when you enter  $\text{SIN } 27$  in degree mode.

4. (Continuation) Repeat the process on a right triangle that has a 10-cm hypotenuse and a 65-degree angle. Try an example of your choosing. Write a summary of your findings.

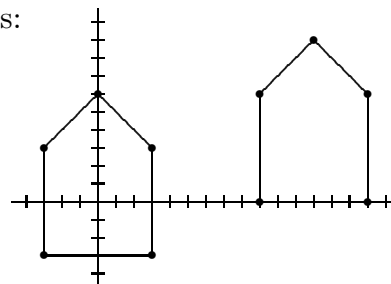
5. If you were to drop a ball from a height of 50 feet, how high would it bounce? To make such a prediction, you could gather data by experimenting with smaller heights, where it is easier to measure the rebound. Gather several data points (drop, rebound), using a meter stick and a sufficiently bouncy ball. If you use the top of the ball for your measurements, remember to take the diameter of the ball into account when recording your data.

6. Apply  $\mathcal{T}(x, y) = (2x/3, 2y/3)$  to the following pentagons:

(a) vertices  $(3, -3)$ ,  $(3, 3)$ ,  $(0, 6)$ ,  $(-3, 3)$ , and  $(-3, -3)$ ;

(b) vertices  $(15, 0)$ ,  $(15, 6)$ ,  $(12, 9)$ ,  $(9, 6)$ , and  $(9, 0)$ .

Are the results what you expected?



7. The area of a parallelogram can be found by multiplying the distance between two parallel sides by the length of either of those sides. Explain why this formula works.

8. The area of a trapezoid can be found by multiplying its altitude (the distance between the parallel sides) by the length of its midline. Explain why this formula works. One approach is to find a suitable rectangle that has the same area as the trapezoid.

9. The parallel sides of a trapezoid have lengths 9 cm and 12 cm. Draw *one* diagonal, dividing the trapezoid into two triangles. What is the ratio of their areas? If the other diagonal had been drawn instead, would this have affected your answer?

10. If triangle  $ABC$  has a right angle at  $C$ , the ratio  $AC : AB$  is called the *sine ratio of angle B*, or simply the *sine of B*, and is usually written  $\sin B$ . What should the ratio  $BC : AB$  be called? Without using your calculator, can you predict what the value of the sine ratio for a 30-degree angle is? How about the sine ratio for a 60-degree angle?

11. Find a vector of length 3 that is perpendicular to (a)  $[2, 1, -2]$ ; (b)  $[4, 4, 7]$ .

## Mathematics 2

1. Let  $C = (1, 4)$ ,  $P = (5, 2)$ , and  $P' = (13, -2)$ . There is a dilation that leaves  $C$  where it is and transforms  $P$  into  $P'$ . The point  $C$  is called the *dilation center*. Explain why the *magnitude* of this dilation is 3. Calculate  $Q'$ , given that  $Q = (3, 5)$ . Calculate  $R$ , given that  $R' = (-6, 7)$ .

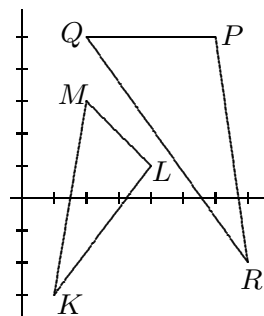
2. Given  $A = (4, 1, 3)$  and  $B = (6, 2, 1)$ , find coordinates for points  $C$  and  $D$  that make  $ABCD$  a square. There are many possible answers.

3. Apply the transformation  $\mathcal{T}(x, y) = (0.8x - 0.6y, 0.6x + 0.8y)$  to the scalene triangle whose vertices are  $(0, 0)$ ,  $(5, 0)$ , and  $(0, 10)$ . What kind of isometry does  $\mathcal{T}$  seem to be? Be as specific as you can, and provide numerical evidence for your conclusion.

4. Consider the dilation  $\mathcal{T}(x, y) = (mx, my)$ , where  $m$  is a positive number. If  $m$  is greater than 1, then  $\mathcal{T}$  is a radial expansion; if  $m$  is smaller than 1, then  $\mathcal{T}$  is a radial contraction. Regardless of the value of  $m$ , show that  $\mathcal{T}$  transforms any segment onto an image segment that is parallel and  $m$  times as long.

5. To actually draw a right triangle that has a 1-degree angle and measure its sides accurately is difficult. To get the sine ratio for a 1-degree angle, however, there is an easy way — just use your calculator. Is the ratio a small or large number? How large can a sine ratio be?

6. One figure is *similar* to a second figure if the points of the first figure can be matched with the points of the second figure in such a way that corresponding distances are proportional. In other words, there is a number  $m$  — the *ratio of similarity* — with the following property: If  $A$  and  $B$  are any two points whatsoever of the first figure, and  $A'$  and  $B'$  are their corresponding images in the second figure, then the distance from  $A'$  to  $B'$  is  $m$  times the distance from  $A$  to  $B$ .



The triangle defined by  $K = (1, -3)$ ,  $L = (4, 1)$ , and  $M = (2, 3)$  is similar to the triangle defined by  $P = (6, 5)$ ,  $Q = (2, 5)$ , and  $R = (7, -2)$ . Confirm the proportionality of lengths for *four* segments (include a non-edge) and their images.

7. Show that any dilation transforms any figure into a similar figure.

8. Given two similar figures, it might not be possible to transform one into the other using only a dilation. Explain this remark, using the triangles  $KLM$  and  $RPQ$  shown above.

9. (Continuation) A carefully chosen rotation followed by a carefully chosen dilation can be used to transform triangle  $KLM$  into triangle  $RPQ$ . Explain this remark.

10. A rhombus has four 6-inch sides and two 120-degree angles. From one of the vertices of the obtuse angles, the two altitudes are drawn, dividing the rhombus into three pieces. Find the areas of these pieces.

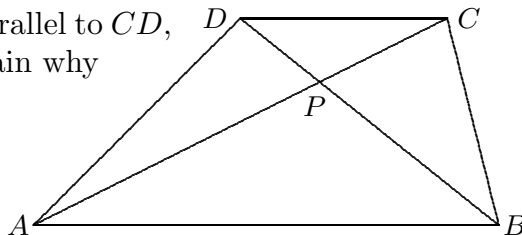
## Mathematics 2

1. When triangle  $ABC$  is similar to triangle  $PQR$ , with  $A$ ,  $B$ , and  $C$  corresponding to  $P$ ,  $Q$ , and  $R$ , respectively, it is customary to write  $ABC \sim PQR$ . Suppose that  $AB = 4$ ,  $BC = 5$ ,  $CA = 6$ , and  $RP = 9$ . Find  $PQ$  and  $QR$ .
2. What is the size of the acute angle formed by the  $x$ -axis and the line  $3x + 2y = 12$ ?
3. To the nearest tenth of a degree, find the sizes of the acute angles in the right triangle whose long leg is 2.5 times as long as its short leg.
4. Draw a large acute-angled triangle  $ABC$ . Carefully add the altitudes  $AE$  and  $BF$  to your drawing. Measure the lengths of  $AE$ ,  $BF$ ,  $BC$ , and  $AC$ . Where have you seen the equation  $(AE)(BC) = (BF)(AC)$  before? What can you say about the right triangles  $AEC$  and  $BFC$ ? Justify your response.
5. Let  $A = (0, 0)$ ,  $B = (15, 0)$ ,  $C = (5, 8)$ ,  $D = (9, 0)$ , and  $P = (6, 6)$ . Draw triangle  $ABC$ , segments  $CD$ ,  $PA$ , and  $PB$ , and notice that  $P$  is on segment  $CD$ . There are now three pairs of triangles in the figure whose areas are in a 3:2 ratio. Find them, and justify your choices.
6. The transformation  $\mathcal{T}(x, y) = (ax + by, cx + dy)$  sends  $(13, 0)$  to  $(12, 5)$  and it also sends  $(0, 1)$  to  $(-\frac{5}{13}, \frac{12}{13})$ . Find  $a$ ,  $b$ ,  $c$ , and  $d$ , then describe the nature of this transformation.
7. Let  $A = (0, 5)$ ,  $B = (-2, 1)$ ,  $C = (6, -1)$ , and  $P = (12, 9)$ . Let  $A'$ ,  $B'$ ,  $C'$  be the midpoints of segments  $PA$ ,  $PB$ , and  $PC$ , respectively. After you make a diagram, identify the center and the magnitude of the dilation that transforms triangle  $ABC$  onto  $A'B'C'$ .
8. One triangle has sides that are 5 cm, 7 cm, and 8 cm long; the longest side of a similar triangle is 6 cm long. How long are the other two sides?
9. Dale is driving along a highway that is climbing a steady 9-degree slope. After driving for two miles along this road, how much altitude has Dale gained? (One mile = 5280 feet.)
10. (Continuation) How far must Dale travel in order to gain a mile of altitude?
11. Show that  $P = (3.2, 6.3)$  is not on the line  $4x - y = 7$ . Explain how you can tell whether  $P$  is above or below the line.
12. Explain why corresponding angles of similar polygons are necessarily the same size.
13. (Continuation) If all the angles of a triangle are equal in size to the angles of another triangle, the triangles are similar. Justify this statement. Is this the converse of the preceding?
14. One stick is twice as long as another. You break the longer stick at a random point. Now you have three sticks. What is the *probability* that they form a triangle?

## Mathematics 2

1. Suppose that  $ABCD$  is a trapezoid, with  $AB$  parallel to  $CD$ , and diagonals  $AC$  and  $BD$  intersecting at  $P$ . Explain why

- (a) triangles  $ABC$  and  $ABD$  have the same area;
- (b) triangles  $BCP$  and  $DAP$  have the same area;
- (c) triangles  $ABP$  and  $CDP$  are similar;
- (d) triangles  $BCP$  and  $DAP$  need not be similar.



2. Find the size of the acute angle formed by the intersecting lines  $3x + 2y = 12$  and  $x - 2y = -2$ , to the nearest tenth of a degree. Do you need to find the intersection point?

3. Let  $A = (0, 5, 0)$ ,  $B = (-2, 1, 0)$ ,  $C = (6, -1, 0)$ , and  $P = (2, 3, 8)$ . Let  $A'$ ,  $B'$ ,  $C'$  be the midpoints of segments  $PA$ ,  $PB$ , and  $PC$ , respectively. Make a diagram, and describe the relationship between triangle  $ABC$  and its image  $A'B'C'$ .

4. Write an equation that says that  $P = (x, y)$  is 5 units from  $(0, 0)$ . Plot several such points. What is the configuration of all such points called? How many are lattice points?

5. The midpoints of the sides of a quadrilateral are joined to form a new quadrilateral. For the new quadrilateral to be a rectangle, what must be true of the original quadrilateral?

6. Given the line whose equation is  $y = 2x + 3$  and the points  $A = (0, 0)$ ,  $B = (1, 9)$ ,  $C = (2, 8)$ ,  $D = (3, 3)$ , and  $E = (4, 10)$ , do the following:

- (a) Plot the line and the points on the same axes.
- (b) Let  $A'$  be the point on the line that has the same  $x$ -coordinate as  $A$ . Subtract the  $y$ -coordinate of  $A'$  from the  $y$ -coordinate of  $A$ . The result is called the *residual* of  $A$ .
- (c) Calculate the other four residuals.
- (d) What does a residual tell you about the relation between a point and the line?

7. The area of an equilateral triangle with  $m$ -inch sides is 8 square inches. What is the area of a regular hexagon that has  $m$ -inch sides?

8. A parallelogram has 10-inch and 18-inch sides and an area of 144 square inches.

- (a) How far apart are the 18-inch sides?
- (b) How far apart are the 10-inch sides?
- (c) What are the angles of the parallelogram?
- (d) How long are the diagonals?

9. Let  $P = (1.35, 4.26)$ ,  $Q = (5.81, 5.76)$ ,  $R = (19.63, 9.71)$ , and  $R' = (19.63, y)$ , where  $R'$  is on the line through  $P$  and  $Q$ . Calculate the residual value  $9.71 - y$ .

10. (Continuation)

- (a) Given that  $Q' = (5.81, y)$  is on the line through  $P$  and  $R$ , find  $y$ . Calculate  $5.76 - y$ .
- (b) Given that  $P' = (1.35, y)$  is on the line through  $Q$  and  $R$ , find  $y$ . Calculate  $4.26 - y$ .
- (c) Which of the three lines best fits the given data? Why do you think so?

11. Write an equation that describes all the points on the circle whose *center* is at the origin and whose *radius* is (a) 13; (b) 6; (c)  $r$ .

## Mathematics 2

1. Campbell is about to attempt a 30-foot putt on a level surface. The hole is 4 inches in diameter. Remembering the advice of a golf pro, Campbell aims for a mark that is 6 inches from the ball and on the line from the center of the hole to the center of the ball. Campbell misses the mark by a sixteenth of an inch. Does the ball go in the hole?
2. A trapezoid has 11-inch and 25-inch parallel sides, and an area of 216 square inches.
  - (a) How far apart are the parallel sides?
  - (b) If one of the non-parallel sides is 13 inches long, how long is the other one? (N.B. There are two answers to this question. It is best to make a separate diagram for each.)
3. Graph the circle whose equation is  $x^2 + y^2 = 64$ . What is its radius? What do the equations  $x^2 + y^2 = 1$ ,  $x^2 + y^2 = 40$ , and  $x^2 + y^2 = k$  all have in common? How do they differ?
4. Taylor lets out 120 meters of kite string, then wonders how high the kite has risen. Taylor is able to calculate the answer, after using a protractor to measure the 63-degree angle of elevation that the string makes with the ground. How high is the kite, to the nearest meter? What (unrealistic) assumptions did you make in answering this question?
5. Find the sine of a 45-degree angle. Use your calculator *only to check your answer*.
6. A triangle that has a 5-inch and a 6-inch side can be similar to a triangle that has a 4-inch and an 8-inch side. Find an example. Check that your example really *is* a triangle.
7. Let  $A = (1, 5)$ ,  $B = (3, 1)$ ,  $C = (5, 4)$ ,  $A' = (5, 9)$ ,  $B' = (11, -3)$ , and  $C' = (17, 6)$ . Show that there is a dilation that transforms triangle  $ABC$  onto triangle  $A'B'C'$ . In other words, find the dilation center and the magnification factor.
8. (Continuation) Calculate the areas of triangles  $ABC$  and  $A'B'C'$ . Are your answers related in a predictable way?
9. The vectors  $[8, 0]$  and  $[3, 4]$  form a parallelogram. Find the lengths of its altitudes.
10. The vertices of triangle  $ABC$  are  $A = (-5, -12)$ ,  $B = (5, -12)$ , and  $C = (5, 12)$ . Confirm that the circumcenter of  $ABC$  lies at the origin. What is an equation for the circumcircle?
11. If the sides of a triangle are 13, 14, and 15 cm long, then the altitude drawn to the 14-cm side is 12 cm long. How long are the other two altitudes? Which side has the longest altitude?
12. (Continuation) How long are the altitudes of a triangle whose sides are 26, 28, and 30 cm long? What happens to the area of a triangle if its dimensions are doubled?
13. Find the length of the bisector of the smallest angle of a 3-4-5 triangle.

## Mathematics 2

1. Show the lines  $(x, y, z) = (5 + 2t, 3 + 2t, 1 - t)$  and  $(x, y, z) = (13 - 3r, 13 + 2r, 4 + 4r)$  are not parallel, and that they do not intersect. Such lines are called *skew*.
2. Let  $A = (6, 0)$ ,  $B = (0, 8)$ ,  $C = (0, 0)$ . In triangle  $ABC$ , let  $F$  be the *foot* of the altitude drawn from  $C$  to side  $AB$ .
  - (a) Explain why the angles of triangles  $ABC$ ,  $CBF$ , and  $ACF$  are the same.
  - (b) Find coordinates for  $F$ , and use them to calculate the exact lengths  $FA$ ,  $FB$ , and  $FC$ .
  - (c) Compare the sides of triangle  $ABC$  with the sides of triangle  $ACF$ . What do you notice?
3. A *similarity transformation* is a geometric transformation that uniformly multiplies distances, in the following sense: For some positive number  $m$ , and *any* two points  $A$  and  $B$  and their respective images  $A'$  and  $B'$ , the distance from  $A'$  to  $B'$  is  $m$  times the distance from  $A$  to  $B$ . You have recently shown that any dilation  $\mathcal{T}(x, y) = (mx, my)$  is a similarity transformation. Is it true that the transformation  $\mathcal{T}(x, y) = (3x, 2y)$  is a similarity transformation? Explain.
4. The area of the triangle determined by the vectors  $\mathbf{v}$  and  $\mathbf{w}$  is 5. What is the area of the triangle determined by the vectors  $2\mathbf{v}$  and  $3\mathbf{w}$ ? Justify your answer. Do not assume that  $\mathbf{v}$  and  $\mathbf{w}$  are perpendicular.
5. Decide whether the transformation  $\mathcal{T}(x, y) = (4x - 3y, 3x + 4y)$  is a similarity transformation. If so, what is the multiplier  $m$ ?
6. A rectangular sheet of paper is 20.5 cm wide. When it is folded in half, with the crease running parallel to the 20.5-cm sides, the resulting rectangle is the same shape as the unfolded sheet. Find the length of the sheet, to the nearest tenth of a cm. (In Europe, the shape of notebook paper is determined by this similarity property.)
7. How much evidence is needed to be sure that two triangles are similar?
8. A line of slope  $\frac{1}{2}$  intersects a line of slope 3. Find the size of the acute angle that these lines form, to the nearest tenth of a degree.
9. Square  $ABCD$  has 8-inch sides,  $M$  is the midpoint of  $BC$ , and  $N$  is the intersection of  $AM$  and diagonal  $BD$ . Find the lengths of  $NB$ ,  $NM$ ,  $NA$ , and  $ND$ .
10. Parallelogram  $PQRS$  has  $PQ = 8$  cm,  $QR = 9$  cm, and diagonal  $QS = 10$  cm. Mark  $F$  on  $RS$ , exactly 5 cm from  $S$ . Let  $T$  be the intersection of  $PF$  and  $QS$ . Find the lengths  $TS$  and  $TQ$ .
11. The parallel sides of a trapezoid are 12 inches and 18 inches long. The non-parallel sides meet when one is extended 9 inches and the other is extended 16 inches. How long are the non-parallel sides of this trapezoid?

## Mathematics 2

1. The lengths of the sides of triangle  $ABC$  are often abbreviated by writing  $a = BC$ ,  $b = CA$ , and  $c = AB$ . Notice that lower-case sides oppose upper-case vertices. Suppose now that angle  $BCA$  is right, so that  $a^2 + b^2 = c^2$ . Let  $F$  be the foot of the perpendicular drawn from  $C$  to the hypotenuse  $AB$ . In terms of  $a$ ,  $b$ , and  $c$ , express the lengths of  $FA$ ,  $FB$ , and  $FC$ . The equation  $c = FA + FB$  can be used to check your work.
2. Verify that  $P = (-1.15, 0.97)$ ,  $Q = (3.22, 2.75)$ , and  $R = (9.21, 10.68)$  are not collinear.
  - (a) Let  $Q' = (3.22, y)$  be the point on the line through  $P$  and  $R$  that has the same  $x$ -coordinate as  $Q$  has. Find  $y$ , then calculate the *residual* value  $2.75 - y$ .
  - (b) Because the segment  $PR$  seems to provide the most accurate slope, one might regard  $PR$  as the line that best fits the given data. The point  $Q$  has as yet played no part in this decision, however. Find an equation for the line that is parallel to  $PR$  and that makes the sum of the three residuals zero. In this sense, this is the *line of best fit*.
3. Apply the Angle-Bisector Theorem to the smallest angle of the right triangle whose sides are 1, 2, and  $\sqrt{3}$ . The side of length 1 is divided by the bisector into segments of what lengths? Check your answer by asking your calculator for the tangent of a 15-degree angle.
4. Sketch the circle whose equation is  $x^2 + y^2 = 100$ . Using the same system of coordinate axes, graph the line  $x + 3y = 10$ , which should intersect the circle twice — at  $A = (10, 0)$  and at another point  $B$  in the second quadrant. Estimate the coordinates of  $B$ . Now use algebra to find them exactly. Segment  $AB$  is called a *chord* of the circle.
5. (Continuation) What percentage of the circumference of the circle lies above the chord  $AB$ ? First estimate the percentage, then find a way of calculating it precisely.
6. (Continuation) Find coordinates for a point  $C$  on the circle that makes chords  $AB$  and  $AC$  have equal length. What percentage of the circumference lies below chord  $AC$ ?
7. What is the radius of the smallest circle that surrounds a 5-by-12 rectangle?
8. A triangle is defined by placing vectors  $[5, 7]$  and  $[-21, 15]$  tail to tail. Find its angles.
9. Without doing any calculation, what can you say about the tangent of a  $k$ -degree angle, when  $k$  is a number between 90 and 180? Explain your response, then check with your calculator.
10. Draw a right triangle that has an 18-cm hypotenuse and a 70-degree angle. To within 0.1 cm, measure the leg *adjacent* to the 70-degree angle, and express your answer as a percentage of the hypotenuse. Compare your answer with the value obtained from your calculator when you enter  $\text{COS } 70$  in degree mode. This is an example of the *cosine ratio*.



## Mathematics 2

1. What is the radius of the largest circle that you can draw on graph paper that encloses
  - (a) *no* lattice points?
  - (b) exactly *one* lattice point?
  - (c) exactly *two* lattice points?
  - (d) exactly *three* lattice points?
2. Let  $\lambda$  be the line  $y = 1$  and  $F$  be the point  $(-1, 2)$ . Verify that the point  $(2, 6)$  is equidistant from  $\lambda$  and  $F$ . Sketch the configuration of *all* points  $P$  that are equidistant from  $F$  and  $\lambda$ . Recall that this curve is called a *parabola*. Point  $F$  is called its *focus*, and line  $\lambda$  is called its *directrix*. Find an equation that says that  $P = (x, y)$  is on the parabola.
3. (Continuation) Let  $N = (2, 1)$ , and find an equation for the perpendicular bisector of  $FN$ . As a check, verify that  $P = (2, 6)$  is on this line. (Why could this have been predicted?) Explain why this line intersects the parabola only at  $P$ .
4. The sides of a triangle are 12 cm, 35 cm, and 37 cm long.
  - (a) Show that this is a right triangle.
  - (b) Show that  $\text{TAN}^{-1}$ ,  $\text{SIN}^{-1}$ , and  $\text{COS}^{-1}$  can all be used to find the size of the smallest angle of this triangle.
5. Suppose that one of the angles of a triangle is exactly twice the size of another angle of the triangle. Show that *any* such triangle can be dissected, by a single straight cut, into two triangles, one of which is isosceles, the other of which is similar to the original.
6. The line  $y = x + 2$  intersects the circle  $x^2 + y^2 = 10$  in two points. Call the third-quadrant point  $R$  and the first-quadrant point  $E$ , and find their coordinates. Let  $D$  be the point where the line through  $R$  and the center of the circle intersects the circle again. The chord  $DR$  is an example of a *diameter*. Show that triangle  $RED$  is a right triangle.
7. (Continuation) The portion of the circle that lies above chord  $RE$  is called an *arc*. Find a way of calculating and describing its *size*. The portion of the circle that lies below line  $RE$  is also an arc — a *major* arc because it is more than half the circle. The part of the circle that lies above chord  $DE$  is a *minor* arc. Because both of the arcs determined by diameter  $DR$  are *semicircles*, it is straightforward to find the size of minor arc  $DE$  once you know the size of minor arc  $RE$ . Explain how to do it.
8. For their students who turn the steering wheel too often while on the freeway, driving instructors suggest that it is better to focus on a point that is about 100 yards ahead of the car than to focus on a point only 10 yards ahead of the car. Comment on this advice.
9. Calculate the residual of  $P = (1.2, 2.4)$  with respect to the line  $3x + 4y = 12$ .
10. Let  $A = (0, 0)$ ,  $B = (120, 160)$ , and  $C = (-75, 225)$ , and let the altitudes of triangle  $ABC$  be segments  $AD$ ,  $BE$ , and  $CF$ , where  $D$ ,  $E$ , and  $F$  are on the sides of the triangle.
  - (a) Show that  $(AF)(BD)(CE) = (FB)(DC)(EA)$ .
  - (b) Show that this equation is in fact valid for *any* acute triangle  $ABC$ . (*Hint*: One way to proceed is to divide both sides of the proposed equation by  $(AB)(BC)(CA)$ .)

## Mathematics 2

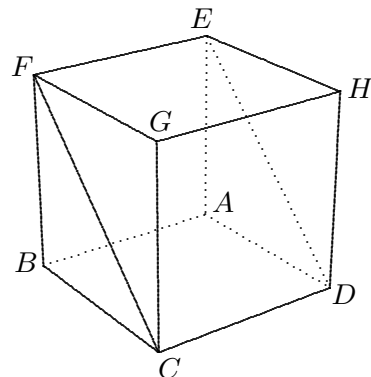
1. Show that the area of a square is half the product of its diagonals. Then consider the possibility that there might be other quadrilaterals with the same property.
2. Let  $A$  and  $B$  be the positive  $x$ -intercept and the positive  $y$ -intercept, respectively, of the circle  $x^2 + y^2 = 18$ . Let  $P$  and  $Q$  be the positive  $x$ -intercept and the positive  $y$ -intercept, respectively, of the circle  $x^2 + y^2 = 64$ . Verify that the ratio of chords  $AB:PQ$  matches the ratio of the corresponding diameters. What does this data suggest to you?
3. Ask your calculator for the sine of a 56-degree angle, and for the cosine of a 34-degree angle. Ask your calculator for the sine of a 23-degree angle, and for the cosine of a 67-degree angle. The word *cosine* abbreviates “sine of the complement.” Explain the terminology. The cosine button on your calculator seems to be unnecessary, and yet it is there. Explain.
4. To the nearest tenth of a degree, find the angles of the triangle with vertices  $(0,0)$ ,  $(6,3)$ , and  $(1,8)$ . Use your protractor to *check* your calculations, and explain your method.
5. In a right triangle, the 58-cm hypotenuse makes a 51-degree angle with one of the legs. To the nearest tenth of a cm, how long is that leg? Once you have the answer, find two ways to calculate the length of the other leg. They should both give the same answer.
6. Make an accurate drawing of a regular hexagon  $ABCDEF$ . Be prepared to report on the method you used to draw this figure. Measure the length of diagonal  $AC$  and the length of side  $AB$ . Form the *ratio* of the diagonal measurement to the side measurement. When you compare answers with your classmates, on which of these three numbers do you expect to find agreement?
7. (Continuation) Calculate  $AC:AB$ , which is the ratio of the diagonal length to the side length in any regular hexagon. One way to do it is to use trigonometry.
8. (Continuation) The diagonals  $AC$ ,  $BD$ ,  $CE$ ,  $DF$ ,  $EA$ , and  $FB$  form the familiar six-pointed *Star of David*. Their intersections inside  $ABCDEF$  are the vertices of a smaller hexagon. Explain why the small hexagon is similar to  $ABCDEF$ . In particular, explain why the small hexagon is regular. Make measurements and use them to approximate the *ratio of similarity*. Then calculate an exact value for this ratio.
9. Given  $T = (1.20, 7.48)$ ,  $U = (4.40, 6.12)$ , and  $V = (8.80, 2.54)$ , find an equation for the line that is parallel to the line  $TV$  and that makes the sum of the three residuals zero. This line is called the *zero-residual line* determined by  $T$ ,  $U$ , and  $V$ .
10. The sides of a square are parallel to the coordinate axes. Its vertices lie on the circle of radius 5 whose center is at the origin. Find coordinates for the four vertices of this square.
11. Find the lengths of both altitudes in the parallelogram determined by  $[2, 3]$  and  $[-5, 7]$ .

## Mathematics 2

1. Let  $A = (0, 1)$ ,  $B = (7, 0)$ ,  $C = (3, 7)$ , and  $D = (0, 6)$ . Find the areas of triangles  $ABC$  and  $ADC$ , which share side  $AC$ . Calculate the ratio of areas  $ABC:ADC$ . If you were to calculate the distances from  $B$  and  $D$  to the line  $AC$ , how would they compare? Explain your reasoning, or else calculate the two distances to confirm your prediction.

2. Draw a circle and label one of its diameters  $AB$ . Choose any other point on the circle and call it  $C$ . What can you say about the size of angle  $ACB$ ? Does it depend on which  $C$  you chose? Justify your response.

3. The figure at right shows a cube  $ABCDEFGH$ . Square  $ABCD$  and rectangle  $EFCD$  form an angle that is called *dihedral* because it is formed by two intersecting planes. The line of intersection here is  $CD$ . Calculate the size of this angle.



4. Draw a large triangle  $ABC$ , and mark  $D$  on segment  $AC$  so that the ratio  $AD:DC$  is equal to 3:4. Mark any point  $P$  on segment  $BD$ .

- (a) Find the ratio of the area of triangle  $BAD$  to the area of triangle  $BCD$ .
- (b) Find the ratio of the area of triangle  $PAD$  to the area of triangle  $PCD$ .
- (c) Find the ratio of the area of triangle  $BAP$  to the area of triangle  $BCP$ .

5. Suppose that triangle  $ABC$  has a 30-degree angle at  $A$  and a 60-degree angle at  $B$ . Let  $O$  be the midpoint of  $AB$ . Draw the circle centered at  $O$  that goes through  $A$ . Explain why this circle also goes through  $B$  and  $C$ . Angle  $BOC$  is called a *central angle* of the circle because its vertex is at the center. The minor arc  $BC$  is called a *60-degree arc* because it *subtends* a 60-degree angle at the center. What is the *angular size* of minor arc  $AC$ ? of *major arc AC*? How does the actual length of minor arc  $AC$  compare to the length of minor arc  $BC$ ?

6. A triangle has two  $k$ -inch sides that make a 36-degree angle, and the third side is one inch long. Draw the bisector of one of the other angles. How long is it? There are several ways to calculate the number  $k$ . Apply at least two of them.

7. Let  $A = (0, 0)$ ,  $B = (12, 0)$ ,  $C = (8, 6)$ , and  $D = (2, 6)$ . The diagonals  $AC$  and  $BD$  of trapezoid  $ABCD$  intersect at  $P$ . Explain why triangle  $ABP$  is similar to triangle  $CDP$ . What is the ratio of similarity? Which side of triangle  $CDP$  corresponds to side  $AP$  in triangle  $ABP$ ? Why is it inaccurate to write  $ABP \sim DCP$ ? Without finding the coordinates of  $P$ , show how you can find the lengths  $AP$  and  $PC$ .

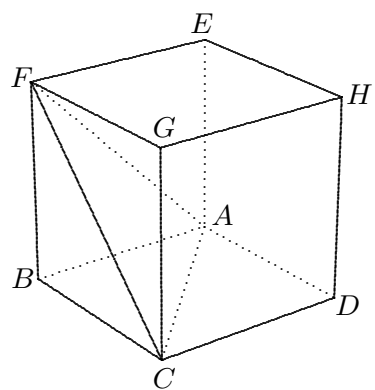
8. (Continuation) Find the ratio of the areas of triangles

- (a)  $ADP$  to  $CDP$ ;      (b)  $ADP$  to  $ABP$ ;      (c)  $CDP$  to  $ABP$ .

9. Consider the points  $A = (-0.5, -8)$ ,  $B = (0.5, -5)$ , and  $C = (3, 4.5)$ . Calculate the residual for each of these points with respect to the line  $4x - y = 7$ .

## Mathematics 2

1. Draw an accurate version of a regular pentagon. Be prepared to report on the method you used to draw this figure. Measure the length of a diagonal and the length of a side. Then divide the diagonal length by the side length. When you and your classmates compare answers, on which of the preceding numbers should you agree — the lengths or the ratio?
2. (Continuation) Calculate the ratio of the diagonal length to the side length in any regular pentagon. One way to do it is to use trigonometry.
3. (Continuation) Label your pentagon  $ABCDE$ . Draw its diagonals. They intersect to form a smaller pentagon  $A'B'C'D'E'$  that lies inside  $ABCDE$ .
  - (a) Explain why  $A'B'C'D'E'$  is regular, and why it is similar to  $ABCDE$ .
  - (b) Measure the length  $A'B'$ , and divide it by  $AB$ . Then use trigonometry to find an exact value for  $A'B' : AB$ , which is called the *ratio of similarity*.
  - (c) Consider the ways of assigning the labels  $A'$ ,  $B'$ ,  $C'$ ,  $D'$ , and  $E'$  to the vertices of the small pentagon. How many ways are there? Is there one that stands out from the rest?
4. (Continuation) It should be possible to *circumscribe* a circle around your pentagon  $ABCDE$ , meaning that a circle can be drawn that goes through all five of its vertices. Find the center of this circle, and describe your method. Then measure the radius of the circle, and express your answer as a multiple of the length  $AB$ . Which of these numbers will be more useful to bring to class — the radius or the ratio?
5. If two chords of a circle have the same length, then their minor arcs have the same length too. True or false? Explain. What about the converse statement? Is it true? Why?
6. The figure at right shows a cube  $ABCDEFGH$ . Triangles  $ABC$  and  $AFC$  form an angle that is called *dihedral* because it is formed by two intersecting planes. Notice that the line of intersection is  $AC$ . Calculate the size of this angle, to the nearest tenth of a degree.
7. Draw a line  $\lambda$  in your notebook, and mark a point  $F$  approximately an inch away from  $\lambda$ . Sketch the parabola that has  $\lambda$  as its directrix and  $F$  as its focus. Find a way of locating that point  $V$  on the parabola that is *closest* to the focus;  $V$  is usually called the *vertex*. Draw the line through  $F$  that is perpendicular to  $\lambda$ . How is this line related to  $V$  and to the parabola?
8. Suppose that  $MP$  is a diameter of a circle centered at  $O$ , and  $Q$  is any other point on the circle. Draw the line through  $O$  that is parallel to  $MQ$ , and let  $R$  be the point where it meets minor arc  $PQ$ . Prove that  $R$  is the midpoint of minor arc  $PQ$ .
9. Line  $\mu$  (Greek “mu”) intersects segment  $AB$  at  $D$ , forming a 57-degree angle. Suppose that  $AD : DB = 2 : 3$  is known. What can you say about the distances from  $A$  to  $\mu$  and from  $B$  to  $\mu$ ? If  $2 : 3$  is replaced by another ratio  $m : n$ , how is your answer affected?

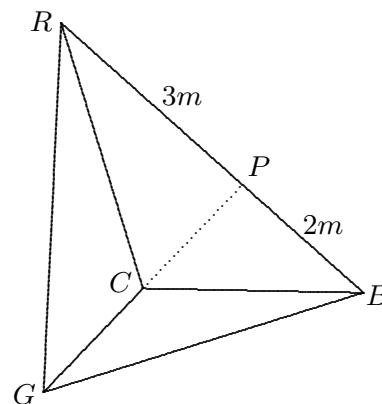


## Mathematics 2

1. The circle  $x^2 + y^2 = 25$  goes through  $A = (5, 0)$  and  $B = (3, 4)$ . To the nearest tenth of a degree, find the size of the minor arc  $AB$ .
2. An equilateral triangle is inscribed in the circle of radius 1 centered at the origin (the *unit circle*). If one of the vertices is  $(1, 0)$ , what are the coordinates of the other two? The three points divide the circle into three arcs; what are the angular sizes of these arcs?
3. On a circle whose center is  $O$ , mark points  $P$  and  $A$  so that minor arc  $PA$  is a 46-degree arc. What does this tell you about angle  $POA$ ? Extend  $PO$  to meet the circle again at  $T$ . What is the size of angle  $PTA$ ? This angle is *inscribed* in the circle, because all three points are on the circle. The arc  $PA$  is *intercepted* by the angle  $PTA$ .
4. (Continuation) If minor arc  $PA$  is a  $k$ -degree arc, what is the size of angle  $PTA$ ?
5. The area of triangle  $ABC$  is 231 square inches, and point  $P$  is marked on side  $AB$  so that  $AP:PB = 3:4$ . What are the areas of triangles  $APC$  and  $BPC$ ?
6. Show that the zero-residual line of the points  $P$ ,  $Q$ , and  $R$  goes through their centroid.
7. (Continuation) The zero-residual line makes the sum of the residuals zero. What about the sum of the *absolute values* of the residuals? Is it possible for this sum to be zero? If not, does the zero-residual line make this sum as small as it can be?
8. Show that the medians of any triangle divide the triangle into six smaller triangles of equal area. Are any of the small triangles necessarily congruent to each other?
9. A close look at a color television screen reveals an array of thousands of tiny red, green, and blue dots. This is because any color can be obtained as a *mixture* of these three colors. For example, if neighboring red, green, and blue dots are equally bright, the effect is white. If a blue dot is unilluminated and its red and green neighbors are equally bright, the effect is yellow. In other words, white corresponds to the red:green:blue ratio  $\frac{1}{3}:\frac{1}{3}:\frac{1}{3}$  and pure yellow corresponds to  $\frac{1}{2}:\frac{1}{2}:0$ . Notice that the sum of the three terms in each proportion is 1. A triangle  $RGB$  provides a simple model for this mixing of colors. The vertices represent three neighboring dots. Each point  $C$  inside the triangle represents a precise color, defined as follows: The *intensities* of the red dot, green dot, and blue dot are proportional to the *areas* of the triangles  $CGB$ ,  $CBR$ , and  $CRG$ , respectively. What color is represented by the centroid of  $RGB$ ? What color is represented by the midpoint of side  $RG$ ?
10. (Continuation) Point  $C$  is  $\frac{3}{5}$  of the way from  $R$  to  $G$ . Give a numerical description for the color mixture that corresponds to it. The color *magenta* is composed of equal intensities of red and blue, with green absent. Where is this color in the triangle?
11. (Continuation) Given that color  $C$  is defined by the red:green:blue ratio  $0.4:g:b$ , where  $g+b=0.6$ , what are the possible positions for  $C$  in the triangle?

## Mathematics 2

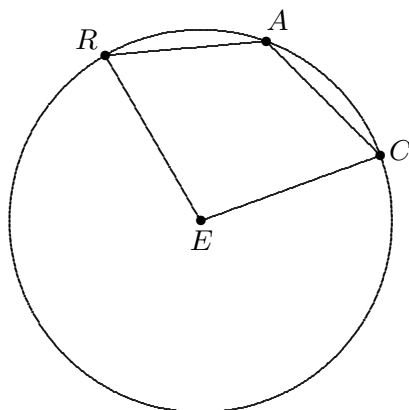
1. Draw non-parallel vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{u} + \mathbf{v}$  emanating from a common point. In order that  $\mathbf{u} + \mathbf{v}$  bisect the angle formed by  $\mathbf{u}$  and  $\mathbf{v}$ , what must be true of  $\mathbf{u}$  and  $\mathbf{v}$ ?
2. The area of a trapezoidal cornfield  $IOWA$  is 18000 sq m. The 100-meter side  $IO$  is parallel to the 150-meter side  $WA$ . This field is divided into four sections by diagonal roads  $IW$  and  $OA$ . Find the areas of the triangular sections.
3. In triangle  $ABC$ , it is given that angle  $BCA$  is right. Let  $a = BC$ ,  $b = CA$ , and  $c = AB$ . Using  $a$ ,  $b$ , and  $c$ , express the sine, cosine, and tangent ratios of acute angles  $A$  and  $B$ .
4. The sine of a 38-degree angle is some number  $r$ . Without using your calculator, you should be able to identify the angle size whose cosine is the same number  $r$ .
5. Given SSS information about an isosceles triangle, describe the process you would use to calculate the sizes of its angles.
6. Triangle  $ABC$  has a 53-degree angle at  $A$ , and its circumcenter is at  $K$ . Draw a good picture of this triangle, and measure the size of angle  $BKC$ . Be prepared to describe the process you used to find  $K$ . Measure the angles  $B$  and  $AKC$  of your triangle. Measure angles  $C$  and  $AKB$ . Make a conjecture about arcs intercepted by inscribed angles. Justify your assertion.
7. If  $P$  and  $Q$  are points on a circle, then the center of the circle must be on the perpendicular bisector of chord  $PQ$ . Explain. Which point on the chord is closest to the center? Why?
8. Given that triangle  $ABC$  is similar to triangle  $PQR$ , write the three-term proportion that describes how the six sides of these figures are related.
9. Draw a circle with a 2-inch radius, mark four points randomly (not evenly spaced) on it, and label them consecutively  $G$ ,  $E$ ,  $O$ , and  $M$ . Measure angles  $GEO$  and  $GMO$ . Could you have predicted the result? Name another pair of angles that would have produced the same result.
10. In triangle  $RGB$ , mark  $P$  on side  $RB$  so that  $RP:PB$  equals 3:2. Let  $C$  be the midpoint of  $GP$ . Calculate the ratio of areas  $CGB:CBR:CRG$ . Express your answer
  - (a) so that the sum of the three numbers is 1;
  - (b) so that the three numbers are all integers.
11. Mixtures of *three* quantities can be modeled geometrically by using a triangle. What geometric figure would be suitable for describing the mixing of *two* quantities? the mixing of *four* quantities? Give the details of your models.



## Mathematics 2

1. A circular park 80 meters in diameter has a straight path cutting across it. It is 24 meters from the center of the park to the closest point on this path. How long is the path?
2. Show that the lines  $y = 2x - 5$  and  $-2x + 11y = 25$  create chords of equal length when they intersect the circle  $x^2 + y^2 = 25$ . Make a large diagram, and measure the inscribed angle formed by these chords. Describe two ways of calculating its size to the nearest 0.1 degree. What is the angular size of the arc that is intercepted by this inscribed angle?
3. A triangle has a 3-inch side, a 4-inch side, and a 5-inch side. The altitude drawn to the 5-inch side cuts this side into segments of what lengths?
4. The parallel sides of a trapezoid are 8 inches and 12 inches long, while one of the non-parallel sides is 6 inches long. How far must this side be extended to meet the extension of the opposite side? What are the possible lengths for the opposite side?
5. The midline of a trapezoid is *not* concurrent with the diagonals. Explain why.
6. A chord 6 cm long is 2 cm from the center of a circle. How long is a chord that is 1 cm from the center of the same circle?
7. By using the triangle whose sides have lengths 1,  $\sqrt{3}$ , and 2, you should be able to write non-calculator expressions for the sine, cosine, and tangent of a 30-degree angle. Do so. You can use your calculator to check your answers, of course.
8. Triangle  $ABC$  is inscribed in a circle. Given that  $AB$  is a 40-degree arc and  $ABC$  is a 50-degree angle, find the sizes of the other arcs and angles in the figure.
9. Suppose that chords  $AB$  and  $BC$  have the same lengths as chords  $PQ$  and  $QR$ , respectively, with all six points belonging to the same circle (they are *concylic*). Is this enough information to conclude that chords  $AC$  and  $PR$  have the same length? Explain.

10. The figure at right shows points  $C$ ,  $A$ , and  $R$  marked on a circle centered at  $E$ , so that chords  $CA$  and  $AR$  have the same length, and so that major arc  $CR$  is a 260-degree arc. Find the angles of quadrilateral  $CARE$ . What is special about the sizes of angles  $CAR$  and  $ACE$ ?



11. The sides of a triangle are found to be 10 cm, 14 cm, and 16 cm long, while the sides of another triangle are found to be 15 in, 21 in, and 24 in long. On the basis of this information, what can you say about the angles of these triangles? Is it possible to *calculate* their sizes?
12. The points  $A$ ,  $P$ ,  $Q$ , and  $B$  appear in this order on a line, so that  $AP:PQ = 2:3$  and  $PQ:QB = 5:8$ . Find whole numbers that are proportional to  $AP:PQ:QB$ .

## Mathematics 2

1. A trapezoid has two 65-degree angles, and also 8-inch and 12-inch parallel sides. How long are the non-parallel sides? What is the area enclosed by this figure?
2. The dimensions of rectangle  $ABCD$  are  $AB = 12$  and  $BC = 16$ . Point  $P$  is marked on side  $BC$ , so that  $BP = 5$ , and the intersection of  $AP$  and  $BD$  is called  $T$ . Find the lengths of the four segments  $TA$ ,  $TP$ ,  $TB$ , and  $TD$ .
3. Two circles of radius 10 cm are drawn so that their centers are 12 cm apart. The two points of intersection determine a *common chord*. Find the length of this chord.
4. Let  $P = (2, 6)$ ,  $Q = (8, 10)$ , and  $R = (11, 2)$ . Find an equation for the zero-residual line, as well as the line of slope 2 through the centroid  $G$  of triangle  $PQR$ . Find the sum of the residuals of  $P$ ,  $Q$ , and  $R$  with respect to the second line. Repeat the investigation using the line of slope  $-1$  through  $G$ . Use your results to formulate a conjecture.
5. Find a triangle two of whose angles have sizes  $\text{TAN}^{-1}(1.5)$  and  $\text{TAN}^{-1}(3)$ . Answer this question either by giving coordinates for the three vertices, or by giving the lengths of the three sides. To the nearest 0.1 degree, find the size of the third angle in your triangle.
6. What is the sine of the angle whose tangent is 2? First find an answer *without* using your calculator (draw a picture), then use your calculator to check.
7. Consider the line  $y = 1.8x + 0.7$ .
  - (a) Find a point whose residual with respect to this line is  $-1$ .
  - (b) Describe the configuration of points whose residuals are  $-1$  with respect to this line.
8. The *median* of a set of numbers is the middle number, once the numbers have been arranged in order. If there are two middle numbers, then the median is half their sum. Find the median of (a) 5, 8, 3, 9, 5, 6, 8; (b) 4, 10, 8, 7.
9. A *median-median point* for a set of points is the point whose  $x$ -value is the median of *all* the given  $x$ -values and whose  $y$ -value is the median of *all* the given  $y$ -values. Find the median-median point for the following set of points:  $(1, 2)$ ,  $(2, 1)$ ,  $(3, 5)$ ,  $(6, 4)$ , and  $(10, 7)$ .
10. True or false? The midline of a trapezoid divides the figure into two trapezoids, each similar to the original. Explain.
11. Hilary and Dale leave camp and go for a long hike. After going 7 km due east, they turn and go another 8 km in the direction 60 degrees north of east. They plan to return along a straight path. How far from camp are they at this point? Use an angle to describe the direction that Hilary and Dale should follow to reach their camp.
12. A right triangle has 6-inch, 8-inch, and 10-inch sides. A square can be inscribed in this triangle, with one vertex on each leg and two vertices on the hypotenuse. How long are the sides of the square?



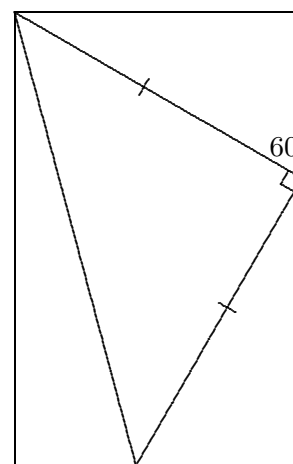
## Mathematics 2

1. In triangle  $RGB$ , point  $X$  divides  $RG$  according to  $RX: XG = 3:5$ , and point  $Y$  divides  $GB$  according to  $GY: YB = 2:7$ . Let  $C$  be the intersection of  $BX$  and  $RY$ .
  - (a) Find a ratio of whole numbers that is equal to the area ratio  $CGB: CBR$ .
  - (b) Find a ratio of whole numbers that is equal to the area ratio  $CBR: CRG$ .
  - (c) Find a ratio of whole numbers that is equal to the area ratio  $CGB: CRG$ .
  - (d) Show that  $CGB: CBR: CRG = m: n: p$ , where  $m$ ,  $n$ , and  $p$  are whole numbers.
  - (e) The line  $GC$  cuts the side  $BR$  into two segments. What is the ratio of their lengths?

2. The area of an equilateral triangle is  $100\sqrt{3}$  square inches. How long are its sides?

3. Points  $P$ ,  $E$ , and  $A$  are marked on a circle whose center is  $R$ . In quadrilateral  $PEAR$ , angles  $A$  and  $E$  are found to be  $54^\circ$  and  $113^\circ$ , respectively. What are the other two angles?

4. The diagram shows a rectangle that has been formed by bordering an isosceles right triangle with three other right triangles, one of which has a 60-degree angle as shown. Find the sizes of the other angles in the figure. By assigning lengths to all the segments, you should be able to work out values for the sine, cosine, and tangent of a 75-degree angle, *without using your calculator's trigonometric functions* (except to check your formulas).



5. The points  $A = (0, 13)$  and  $B = (12, 5)$  lie on a circle whose center is at the origin. Write an equation for the perpendicular bisector of segment  $AB$ . Notice that this bisector goes through the origin; why was this expected?

6. (Continuation) Find center and radius for another circle to which  $A$  and  $B$  both belong, and write an equation for it. How small can such a circle be? How large? What can be said about the centers of all such circles?

7. The areas of two similar triangles are 24 square cm and 54 square cm. The smaller triangle has a 6-cm side. How long is the corresponding side of the larger triangle?

8. When two circles have a common chord, their centers and the endpoints of the chord form a quadrilateral. What kind of quadrilateral is it? What special property do its diagonals have?

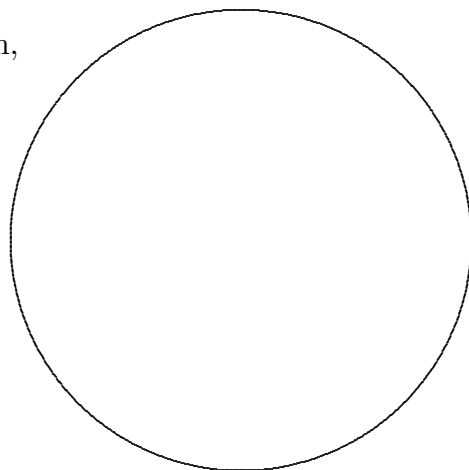
9. The area of triangle  $ABC$  is 75 square cm. Medians  $AN$  and  $CM$  intersect at  $G$ . What is the area of quadrilateral  $GMBN$ ?

10. Given that  $\theta$  (*Greek* “theta”) stands for the degree size of an acute angle, fill in the blank space between the parentheses to create a true statement:  $\sin \theta = \cos ( \quad )$ .

11. If corresponding sides of two similar triangles are in a 3:5 ratio, then what is the ratio of the areas of these triangles?

## Mathematics 2

1. Let  $P = (-25, 0)$ ,  $Q = (25, 0)$ , and  $R = (-24, 7)$ .
  - (a) Find an equation for the circle that goes through  $P$ ,  $Q$ , and  $R$ .
  - (b) Find at least two ways of showing that angle  $PRQ$  is right.
  - (c) Find coordinates for three other points  $R$  that would have made angle  $PRQ$  right.
2. Show that  $(-2, 10)$ ,  $(1, 11)$ ,  $(6, 10)$ , and  $(9, 7)$  are concyclic.
3. Explain how to find the center of the circle shown, using only a pencil and a rectangular sheet of paper.
4. Trapezoid  $ABCD$  has parallel sides  $AB$  and  $CD$ , of lengths 8 and 16, respectively. Diagonals  $AC$  and  $BD$  intersect at  $E$ , and the length of  $AC$  is 15. Find the lengths of  $AE$  and  $EC$ .
5. (Continuation) Through  $E$  draw the line parallel to sides  $AB$  and  $CD$ , and let  $P$  and  $Q$  be its intersections with  $DA$  and  $BC$ , respectively. Find the length of  $PQ$ .
6. Plot the following nine non-collinear points:  
(0.0, 1.0) (1.0, 2.0) (2.0, 2.7) (3.0, 4.0) (4.0, 3.0) (5.0, 4.6) (6.0, 6.2) (7.0, 8.0) (8.0, 8.5)
  - (a) Use your ruler (clear plastic is best) to draw the line that seems to best fit this data.
  - (b) Record the slope and the  $y$ -intercept of your line.
7. (Continuation) Extend the zero-residual-line technique to this data set as follows: First, working left to right, separate the data into three groups of equal size (three points in each group for this example). Next, select the *summary point* for each group by finding its median-median point. Finally, calculate the zero-residual line defined by these three summary points. This line is called the *median-median line*. Sketch this line, and compare it with your estimated line of best fit.
8. (Continuation) If the number of data points is not divisible by three, the three groups cannot have the same number of points. In such cases, it is customary to arrange the group sizes in a symmetric fashion. For instance:
  - (a) Enlarge the data set to include a tenth point,  $(9.0, 9.5)$ , and then separate the ten points into groups, of sizes three, four, and three points, reading from left to right. Calculate the summary points for these three groups.
  - (b) Enlarge the data set again to include an eleventh point,  $(10.0, 10.5)$ . Separate the eleven points into three groups and calculate the three summary points.
9. Let  $A = (0, 0)$ ,  $B = (4, 0)$ , and  $C = (4, 3)$ . Mark point  $D$  so that  $ACD$  is a right angle and  $DAC$  is a 45-degree angle. Find coordinates for  $D$ . Find the tangent of angle  $DAB$ .
10. Find a point on the line  $y = x$  that lies on the parabola whose focus is  $(0, 2)$  and directrix is the  $x$ -axis. Describe the relationship between the line  $y = x$  and the parabola.



## Mathematics 2

1. Two circles have a 24-cm common chord, their centers are 14 cm apart, and the radius of one of the circles is 13 cm. Make an accurate drawing, and find the radius for the second circle in your diagram. Are there other possible answers?

2. *SAS Similarity.* Use your protractor to carefully draw a triangle that has a 5-cm side, a 9-cm side, and whose *included angle* is 40 degrees. Construct a second triangle that has a 10-cm side, an 18-cm side, and whose included angle is also 40 degrees. Measure the remaining parts of these triangles. Could you have anticipated the results? Explain.

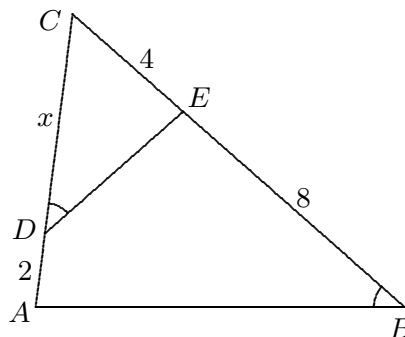
3. Find the perimeter of a regular 36-sided polygon *inscribed* in a circle of radius 20 cm.

4. Find the area of a regular 36-sided polygon inscribed in a circle of radius 20 cm.

5. The position of a starship is given by the equation  $P_t = (18 + 3t, 24 + 4t, 110 - 5t)$ . For what values of  $t$  is the starship within 100 units of a space station placed at the origin?

6. Point  $P = (x, y)$  is 6 units from  $A = (0, 0)$  and 9 units from  $B = (9, 0)$ . Find  $x$  and  $y$ .

7. Refer to the figure, in which angles  $ABE$  and  $CDE$  are equal in size, and various segments have been marked with their lengths. Find  $x$ .



8. Let  $A = (0, 0)$ ,  $B = (7, 0)$ , and  $C = (7, 5)$ . Point  $D$  is located so that angle  $ACD$  is a right angle and the tangent of angle  $DAC$  is  $5/7$ . Find coordinates for  $D$ .

9. A kite has an 8-inch side and a 15-inch side, which form a right angle. Find the length of the diagonals of the kite.

10. Mark points  $A$  and  $C$  on a clean sheet of paper, then spend a minute or so drawing rectangles  $ABCD$ . What do you notice about the configuration of points  $B$  and  $D$ ?

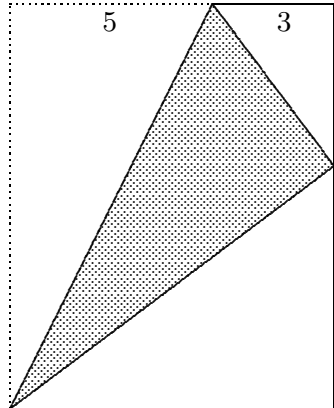
11. What is the radius of the circumscribed circle for a triangle whose sides are 15, 15, and 24 cm long? What is the radius of the smallest circle that contains this triangle?

12. Find an equation for the circle of radius 5 whose center is at  $(3, -1)$ .

13. Draw a *cyclic* quadrilateral  $SPAM$  in which the size of angle  $SPA$  is 110 degrees. What is the size of angle  $AMS$ ? Would your answer change if  $M$  were replaced by a different point on major arc  $SA$ ?

14. Let  $A'B'C'$  be the midpoint triangle of triangle  $ABC$ . In other words,  $A'$ ,  $B'$ , and  $C'$  are the midpoints of segments  $BC$ ,  $CA$ , and  $AB$ , respectively. Show that triangles  $A'B'C'$  and  $ABC$  have the same centroid.

## Mathematics 2

1. Does  $(1, 11)$  lie on the parabola defined by the focus  $(0, 4)$  and the directrix  $y = x$ ? Justify your answer.
2. The area of a trapezoid is 3440 square inches, and the lengths of its parallel sides are in a 3:5 ratio. A diagonal divides the trapezoid into two triangles. What are their areas?
3. Let  $WISH$  be a cyclic quadrilateral, and  $K$  be the intersection of its diagonals  $WS$  and  $HI$ . Given that arc  $WI$  is 100 degrees and arc  $SH$  is 80 degrees, find the sizes of as many angles in the figure as you can.
4. A regular dodecagon can be dissected into regular polygons (which do not all have the same number of sides). Use this dissection (but *not* a calculator) to find the area of the dodecagon, assuming that its edges are all 8 cm long.
5. Let  $A = (0, 0)$  and  $B = (0, 8)$ . Plot several points  $P$  that make  $APB$  a 30-degree angle. Use a protractor, and be prepared to report coordinates for your points. Formulate a conjecture about the configuration of all such points.
6. Triangle  $ABC$  has  $P$  on  $AC$ ,  $Q$  on  $AB$ , and angle  $APQ$  equal to angle  $B$ . The lengths  $AP = 3$ ,  $AQ = 4$ , and  $PC = 5$  are given. Find the length of  $AB$ .
7. The figure at right shows a rectangular sheet of paper that has been creased so that one of its corners matches a point on a non-adjacent edge. Given the dimensions marked on the figure, you are to determine the length of the crease.
 
8. Draw the line  $y = 2x - 5$  and the circle  $x^2 + y^2 = 5$ . Use algebra to show that these graphs touch at only one point. Find the slope of the segment that joins this point to the center of the circle, and compare your answer with the slope of the line  $y = 2x - 5$ . It is customary to say that a line and a circle are *tangent* if they have exactly one point in common.
9. Point by point, the transformation  $\mathcal{T}(x, y) = (4x - y, 3x - 2y)$  sends the line  $x + 2y = 6$  onto an image line. What is the slope of the image?
10. The zero-residual line determined by  $(1, 2)$ ,  $(4, k)$ , and  $(7, 8)$  is  $y = x - \frac{2}{3}$ . Sketch the line, plot the points, and find the value of  $k$ . Be prepared to explain your method.
11. The length of segment  $AB$  is 20 cm. Find the distance from  $C$  to  $AB$ , given that  $C$  is a point on the circle that has  $AB$  as a diameter, and that
  - (a)  $AC = CB$ ;      (b)  $AC = 10$  cm;      (c)  $AC = 12$  cm.
12. Quadrilateral  $BAKE$  is cyclic. Extend  $BA$  to a point  $T$  outside the circle, thus producing the exterior angle  $KAT$ . Why do angles  $KAT$  and  $KEB$  have the same size?

## Mathematics 2

1. A kite has a 5-inch side and a 7-inch side. One of the diagonals is bisected by the other. The bisecting diagonal has length 8 inches. Find the length of the bisected diagonal.

2. Drawn in a circle whose radius is 12 cm, chord  $AB$  is 16 cm long. Calculate the angular size of minor arc  $AB$ .

3. *The reflection property of parabolas.* Consider the parabola whose focus is  $F = (1, 4)$  and whose directrix is the line  $x = -3$ .

(a) Sketch the parabola, and make calculations that confirm that  $P = (7, 12)$  is on it.

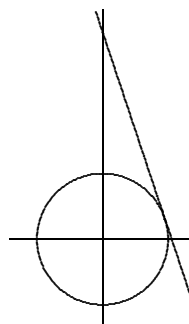
(b) Find the slope of the line  $\mu$  through  $P$  that is tangent to the parabola.

(c) Calculate the size of the angle that  $\mu$  makes with the line  $y = 12$ .

(d) Calculate the size of the angle that  $\mu$  makes with segment  $FP$ . Hmm...

4. The graph of  $x^2 - 6x + 9 + y^2 + 2y + 1 = 25$  is a circle. Where is the center of the circle? What is the radius of the circle?

5. Show that the line  $y = 10 - 3x$  is tangent to the circle  $x^2 + y^2 = 10$ . Find an equation for the line perpendicular to the tangent line at the point of tangency. Show that this line goes through the center of the circle.



6. Let  $K = (0, 0)$ ,  $L = (12, 0)$ , and  $M = (0, 9)$ . Find equations for the three lines that bisect the angles of triangle  $KLM$ . Show that the lines are concurrent at a point  $C$ , the *incenter* of  $KLM$ . Why is  $C$  called this?

7. In triangle  $RGB$ , point  $X$  divides side  $RG$  according to  $RX : XG = m : n$ , and point  $Y$  divides side  $GB$  according to  $GY : YB = p : q$ . Let  $C$  be the intersection of segments  $BX$  and  $RY$ . Find the area ratios

(a)  $CGB : CBR$  (b)  $CBR : CRG$  (c)  $CGB : CRG$  (d)  $CGB : CBR : CRG$

(e) Find the ratio into which the line  $GC$  divides the side  $BR$ .

8. Hanging weights on a spring makes the spring stretch — the greater the mass, the greater the stretch. Some PEA physics students studied a spring in the laboratory. Their seven data points appear at right, measured in kilograms and meters. Find the median-median line and interpret the slope and the intercepts:

mass	stretch
1.960	0.062
3.920	0.125
4.900	0.157
6.860	0.220
9.800	0.317
12.74	0.406
14.70	0.469

9. The altitude drawn to the hypotenuse of a right triangle divides the hypotenuse into two segments, whose lengths are 8 inches and 18 inches. How long is the altitude?

10. Write an equation for the circle that is centered at  $(-4, 5)$  and tangent to the  $x$ -axis.

11. Verify that the point  $A = (8, \frac{25}{3})$  lies on the parabola whose focus is  $(0, 6)$  and whose directrix is the  $x$ -axis. Find an equation for the line that is tangent to the parabola at  $A$ .

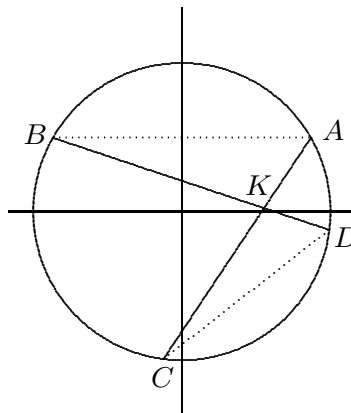
12. Let  $A = (1, 3)$ ,  $B = (6, 0)$ , and  $C = (9, 9)$ . Find the size of angle  $BAC$ . There is more than one way to do it.

## Mathematics 2

1. For (a), find center and radius. For (b), explain why it has the same graph as (a).  
 (a)  $(x - 5)^2 + (y + 3)^2 = 49$       (b)  $x^2 - 10x + y^2 + 6y = 15$
  2. For each of the following, fill in the blank to create a perfect-square trinomial:  
 (a)  $x^2 - 6x + \underline{\hspace{1cm}}$     (b)  $y^2 + 7y + \underline{\hspace{1cm}}$     (c)  $x^2 - 0.4x + \underline{\hspace{1cm}}$     (d)  $y^2 - \underline{\hspace{1cm}}y + 42.25$
  3. Find the center and the radius of the following circles:  
 (a)  $x^2 + y^2 - 6x + y = 3$     (b)  $x^2 + y^2 + 8x = 0$     (c)  $x^2 + y^2 + 2x - 8y = -8$
  4. Let  $K = (5, 12)$ ,  $L = (14, 0)$ , and  $M = (0, 0)$ . The line  $x + 2y = 14$  bisects angle  $MLK$ . Find equations for the bisectors of angles  $KML$  and  $MKL$ . Is the slope of segment  $MK$  twice the slope of the bisector through  $M$ ? Should it have been? Show that the three lines concur at a point  $C$ . Does  $C$  have any special significance?
  5. Trapezoid  $ABCD$  has parallel sides  $AB$  and  $CD$ , of lengths 12 and 18, respectively. Diagonals  $AC$  and  $BD$  intersect at  $E$ . Draw the line through  $E$  that is parallel to  $AB$  and  $CD$ , and let  $P$  and  $Q$  be its intersections with  $DA$  and  $BC$ , respectively. Find  $PQ$ .
- 
6. The point  $P = (4, 3)$  lies on the circle  $x^2 + y^2 = 25$ . Find an equation for the line that is tangent to the circle at  $P$ . This line meets the  $x$ -axis at a point  $Q$ . Find an equation for the other line through  $Q$  that is tangent to the circle, and identify its point of tangency.
  7. Let  $P = (4, 4, 7)$ ,  $A = (0, 0, 0)$ ,  $B = (8, 0, 0)$ ,  $C = (8, 8, 0)$ , and  $D = (0, 8, 0)$ . These points are the vertices of a *regular square pyramid*. Sketch it. To the nearest tenth of a degree, find the size of the dihedral angle formed by *lateral face*  $PCD$  and *base*  $ABCD$ .
  8. (Continuation) Find the size of the angle formed by the edge  $PB$  and the base plane  $ABCD$ . First you will have to decide what this means.
  9. (Continuation) Let  $Q = (5, 5, 7)$ . The five points  $QABCD$  are the vertices of a square pyramid. Explain why the pyramid is *not* regular. To the nearest tenth of a degree, find the size of the dihedral angle formed by the lateral face  $QCD$  and the base  $ABCD$ .
  10. Segment  $AB$ , which is 25 inches long, is the diameter of a circle. Chord  $PQ$  meets  $AB$  perpendicularly at  $C$ , where  $AC = 16$  in. Find the length of  $PQ$ .
  11. How long is the common chord of the circles  $x^2 + y^2 = 4$  and  $x^2 + y^2 = 4x$ ?
  12. Draw the circles  $x^2 + y^2 = 5$  and  $(x - 2)^2 + (y - 6)^2 = 25$  on the same coordinate-axis system. Subtract one equation from the other, and simplify the result. This should produce a linear equation; graph it. Is there anything special about this line? Make a conjecture about what happens when one circle equation is subtracted from another.
  13. Prove that the arcs between any two parallel chords in a circle must be the same size.

## Mathematics 2

1. *Crossed Chords.* Verify that  $A = (7, 4)$ ,  $B = (-7, 4)$ ,  $C = (-1, -8)$ , and  $D = (8, -1)$  all lie on a circle centered at the origin. Let  $K$  be the intersection of chords  $AC$  and  $BD$ . By measuring angles, convince yourself that triangle  $KAB$  is similar to triangle  $KDC$ . Could this have been predicted? Write out the three-term proportion. To three significant figures, what is the ratio of similarity?



2. (Continuation). Explain why triangle  $KAD$  is also similar to triangle  $KBC$ . What is the ratio of similarity? Is it the same ratio as for the other pair of similar triangles?

3. *Two-Tangent Theorem.* From any point  $P$  outside a given circle, there are two lines through  $P$  that are tangent to the circle. Explain why the distance from  $P$  to one of the points of tangency is the same as the distance from  $P$  to the other point of tangency. What special quadrilateral is formed by the center of the circle, the points of tangency, and  $P$ ?

4. A 72-degree arc  $AB$  is drawn in a circle of radius 8 cm. How long is chord  $AB$ ?

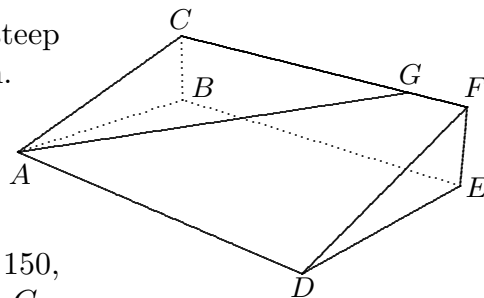
5. Find the perimeter of a regular 360-sided polygon that is inscribed in a circle of radius 5 inches. If someone did not remember the formula for the circumference of a circle, how could that person use a calculator's trigonometric functions to find the circumference of a circle with a 5-inch radius?

6. The line drawn tangent to the circle  $x^2 + y^2 = 169$  at  $(12, 5)$  meets the  $y$ -axis where?

7. The segments  $GA$  and  $GB$  are tangent to a circle at  $A$  and  $B$ , and  $AGB$  is a 60-degree angle. Given that  $GA = 12$  cm, find the distance from  $G$  to the nearest point on the circle.

8. Through the point  $(13, 0)$ , there are two lines that can be drawn tangent to the circle  $x^2 + y^2 = 25$ . Find an equation for one of them. To begin your solution, you could find the common *length* of the tangent segments.

9. Peyton's workout today is to run repeatedly up a steep grassy slope, represented by  $ADFC$  in the diagram. The workout loop is  $AGCA$ , in which  $AG$  requires exertion and  $GCA$  is for recovery. Point  $G$  was chosen on the ridge  $CF$  to make the slope of the climb equal 20%. Given that  $ADEB$  and  $BEFC$  are rectangles,  $ABC$  is a right angle,  $AD = 240$ ,  $DE = 150$ , and  $EF = 50$ , find the distance from point  $G$  to point  $C$ .



10. (Continuation) Peyton's next workout loop is  $AHCA$ , where  $H$  is a point on the path  $AG$ , chosen to make the slope of  $HC$  equal 20%. Find the ratio  $AH/AG$ , and explain your choice.

## Mathematics 2

1. The data shown at right was generated by suspending weights (measured in kilograms) from a rubber band (measured in meters). Find the median-median line and interpret the results. How many meters will the band be stretched by a 4.20-kg weight?

<i>mass</i>	<i>stretch</i>
0.49	0.007
0.68	0.013
0.98	0.019
1.19	0.023
1.47	0.033
1.67	0.039
1.96	0.053
2.94	0.110
3.43	0.144
3.92	0.171
4.41	0.195
4.91	0.259
5.39	0.273

2. A circle goes through the points  $A$ ,  $B$ ,  $C$ , and  $D$  consecutively. The chords  $AC$  and  $BD$  intersect at  $P$ . Given that  $AP = 6$ ,  $BP = 8$ , and  $CP = 3$ , how long is  $DP$ ?

3. Write an equation that says that  $P = (x, y)$  is on the parabola whose focus is  $(2, 1)$  and whose directrix is the line  $y = -1$ .

4. When a regular polygon is inscribed in a circle, the circle is divided into arcs of equal size. The angular size of these arcs is simply related to the size of the interior angles of the polygon. Describe the relationship.

5. A piece of a broken circular gear is brought into a metal shop so that a replacement can be built. A ruler is placed across two points on the rim, and the length of the chord is found to be 14 inches. The distance from the midpoint of this chord to the nearest point on the rim is found to be 4 inches. Find the radius of the original gear.

6. The intersecting circles  $x^2 + y^2 = 100$  and  $(x - 21)^2 + y^2 = 289$  have a common chord. Find its length.

7. (Continuation) The region that is inside both circles is called a *lens*. Find the angular sizes of the two arcs that form the boundary of the lens. Does the common chord of the circles serve as a line of symmetry of the lens?

8. A triangle has two 13-cm sides and a 10-cm side. The largest circle that fits inside this triangle meets each side at a point of tangency. These points of tangency divide the sides of the triangle into segments of what lengths? What is the radius of the circle?

9. A 20-inch chord is drawn in a circle with a 12-inch radius. What is the *angular size* of the minor arc of the chord? What is the *length* of the arc, to the nearest tenth of an inch?

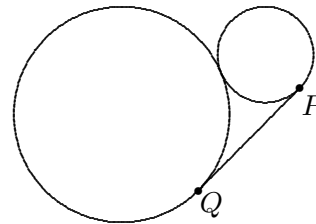
10. A triangle that has a 50-degree angle and a 60-degree angle is inscribed in a circle of radius 25 inches. The circle is divided into three arcs by the vertices of the triangle. To the nearest tenth of an inch, find the lengths of these three arcs.

11. In the Assembly Hall one day, Tyler spends some time trying to figure which row gives the best view of the screen. The screen is 18 feet tall, and its bottom edge is 6 feet above eye level. Tyler finds that sitting 36 feet from the plane of the screen is not satisfactory, for the screen is far away and only subtends a 24.2-degree angle. *Verify this*. Sitting 4 feet from the screen is just as bad, because the screen subtends the same 24.2-degree angle from this position. *Verify this also*. Then find the optimal viewing distance — the distance that makes the screen seem the largest — and the angular size of the screen at this distance.



## Mathematics 2

1. A circle with a 4-inch radius is centered at  $A$ , and a circle with a 9-inch radius is centered at  $B$ , where  $A$  and  $B$  are 13 inches apart. There is a segment that is tangent to the small circle at  $P$  and to the large circle at  $Q$ . It is a common external tangent of the two circles. What kind of quadrilateral is  $PABQ$ ? What are the lengths of its sides?



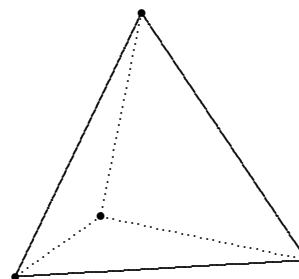
2. Chords  $AB$  and  $CD$  intersect at  $P$ . Given  $AP = 16$ ,  $BP = 9$ , and  $CP = 8$ , find  $DP$ .

3. The data points  $(2.0, 5.5)$ ,  $(8.0, k)$ , and  $(10.0, 1.5)$  determine a median-median line, whose equation is  $y = 6.0 - 0.5x$ . Find  $k$ .

4. Find the radius of the largest circle that can be drawn inside the right triangle that has 6-cm and 8-cm legs.

5. The segments  $GA$  and  $GB$  are tangent to a circle at  $A$  and  $B$ , and  $AGB$  is a 48-degree angle. Given that  $GA = 12$  cm, find the distance from  $G$  to the nearest point on the circle.

6. A regular tetrahedron is a triangular pyramid, all of whose edges have the same length. If all the edges are 6-inch segments, how tall is such a pyramid, to the nearest hundredth of an inch?



7. The line  $x + 2y = 5$  divides the circle  $x^2 + y^2 = 25$  into two arcs. Calculate their lengths. The interior of the circle is divided into two regions by the line. Calculate their areas. Give three significant digits for your answers.

8. Within a given circle, is the length of a circular arc proportional to the length of its chord? Explain your answer.

9. Find an equation for the circle that goes through the points  $(0, 0)$ ,  $(0, 8)$ , and  $(6, 12)$ . Find an equation for the line that is tangent to this circle at  $(6, 12)$ .

10. Can a circle always be drawn through three given points? If so, describe a procedure for finding the center of the circle. If not, explain why not.

11. A dilation  $\mathcal{T}$  sends  $A = (2, 3)$  to  $A' = (5, 4)$ , and it sends  $B = (3, -1)$  to  $B' = (7, -4)$ . Where does it send  $C = (4, 1)$ ? Write a general formula for  $\mathcal{T}(x, y)$ .

12. Find an equation for the line that goes through the two intersection points of the circle  $x^2 + y^2 = 25$  and the circle  $(x - 8)^2 + (y - 4)^2 = 65$ .

13. All triangles and rectangles have circumscribed circles. Is this true for all kites, trapezoids, and parallelograms? Explain. Which quadrilaterals have circumscribed circles? Explain.

## Mathematics 2

1. *An Apollonian circle.* Let  $A = (-5, 0)$  and  $B = (1, 0)$ . Plot a few points  $P = (x, y)$  for which  $PA = 2PB$ , including any that lie on the coordinate axes. Use the distance formula to find an equation for the configuration of all such points. Simplify your equation. Does it help you identify your graph?

2. The figure at right is built by joining six equilateral triangles  $ABC$ ,  $ACD$ ,  $CDE$ ,  $DEF$ ,  $EFG$ , and  $FGH$ , all of whose edges are 1 unit long. It is given that  $HIJKLMB$  is straight.

(a) There are five triangles in the figure that are similar to  $CMB$ . List them, making sure that you match corresponding vertices.

(b) Find the lengths of  $CM$  and  $EK$ .

(c) List the five triangles that are similar to  $AMB$ .

(d) Find the lengths of  $CL$ ,  $HI$ ,  $IJ$ , and  $JK$ .

3. Two of the tangents to a circle meet at  $Q$ , which is 25 cm from the center. The circle has a 7-cm radius. To the nearest tenth of a degree, find the angle formed at  $Q$  by the tangents.

4. To the nearest tenth of a degree, find the angle formed by placing the vectors  $[4, 3]$  and  $[-7, 1]$  tail to tail.

5. Four points on a circle divide it into four arcs, whose sizes are 52 degrees, 116 degrees, 100 degrees, and 92 degrees, in consecutive order. The four points determine two intersecting chords. Find the sizes of the angles formed by the intersecting chords.

6. Let  $A = (3, 4)$  and  $B = (-3, 4)$ , which are both on the circle  $x^2 + y^2 = 25$ .

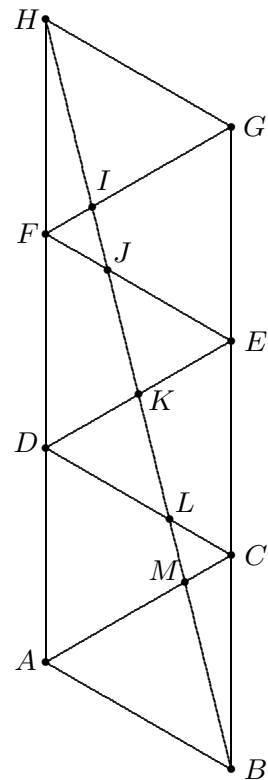
Let  $\lambda$  be the line that is tangent to the circle at  $A$ . Find the angular size of minor arc  $AB$ , then find the size of the acute angle formed by  $\lambda$  and chord  $AB$ . Is there a predictable relation between the two numbers? Explain.

7. What is the radius of the largest circle that will fit inside a triangle that has two 15-inch sides and an 18-inch side?

8. If a line cuts a triangle into two pieces of equal area, must that line go through the centroid of the triangle? Explain your answer.

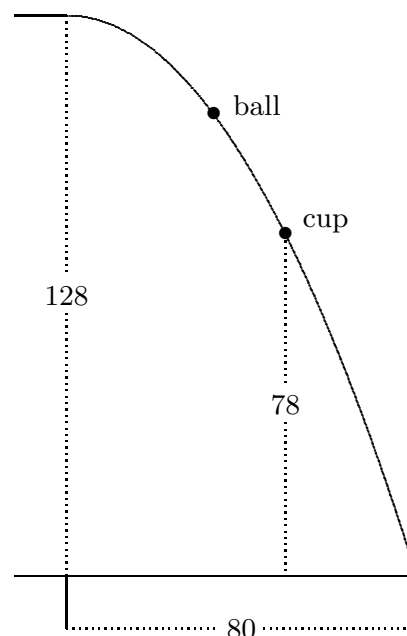
9. What graph is traced by the parametric equation  $(x, y) = (t, 4 - t^2)$ ?

10. Points  $D$  and  $E$  are marked on segments  $AB$  and  $BC$ , respectively. When segments  $CD$  and  $AE$  are drawn, they intersect at point  $T$  inside triangle  $ABC$ . It is found that segment  $AT$  is twice as long as segment  $TE$ , and that segment  $CT$  is twice as long as segment  $TD$ . Must  $T$  be the centroid of triangle  $ABC$ ?



## Mathematics 2

1. After rolling off the end of a ramp, a ball follows a curved trajectory to the floor. To test a theory that says that the trajectory can be described by an equation  $y = h - ax^2$ , Sasha makes some measurements. The end of the ramp is 128 cm above the floor, and the ball lands 80 cm downrange, as shown in the figure. In order to catch the ball in mid-flight with a cup that is 78 cm above the floor, where should Sasha place the cup?



2. Sam and Kirby were out in their rowboat one day, when Kirby spied a nearby water lily. Knowing that Sam liked a mathematical challenge, Kirby announced that, with the help of the plant, it was possible to calculate the depth of the water under the boat. While Sam held the top of the plant, which remained rooted to the lake bottom during the entire process, Kirby gently rowed the boat five feet. This forced Sam's hand to the water surface. When pulled taut, the top of the plant was originally 10 inches above the water surface. Use this information to calculate the depth of the water under the boat.

3. *Crossed Chords Revisited.* Suppose that  $A$ ,  $B$ ,  $D$ , and  $C$  lie (in that order) on a circle, and that chords  $AC$  and  $BD$  intersect, when extended, at a point  $P$  that is *outside* the circle. Explain why  $PA \cdot PC = PB \cdot PD$ .

4. One stick is three times as long as another. You break the longer stick at a random point. Now you have three sticks. What is the probability that they form a triangle?

5. (Continuation) What is the probability that the three pieces form a triangle if the lengths of the original two sticks are  $a$  and  $b$ , and  $a$  is larger than  $b$ ?

6. Commercial jets usually land using a 3-degree angle of descent. At what distance from the destination airport should the pilot begin the descent, if the cruising altitude of the jet is 37000 feet?

7. You have seen that the midline of a trapezoid does *not* divide the trapezoid into two similar trapezoids. Is it possible that a *different* line (parallel to the midline) could divide the the trapezoid into two similar trapezoids?



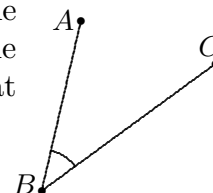
## Mathematics 2 Reference

**AAA similarity:** Two *triangles* are sure to be similar if their angles are equal in size.

**adjacent:** Two vertices of a polygon that are connected by an edge. Two edges of a polygon that intersect at a vertex. Two angles of a polygon that have a common side.

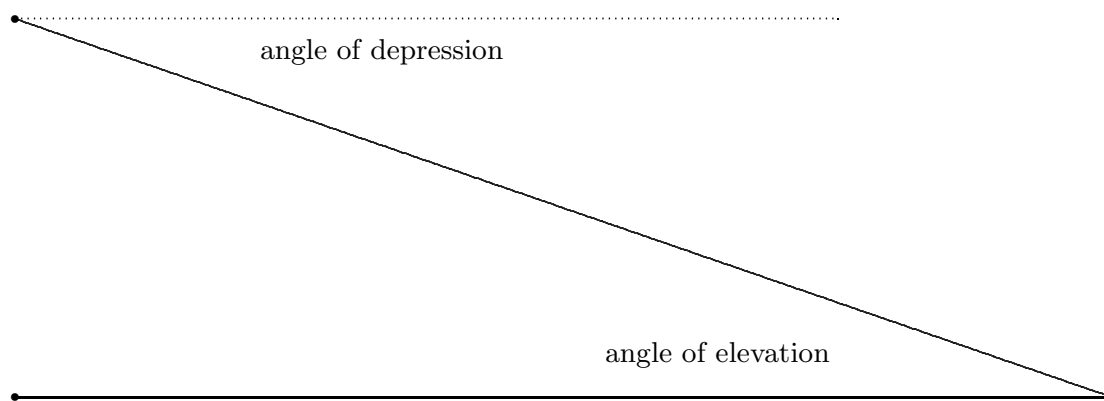
**Alex in the desert:** [12,17,32,35]

**altitude:** In a triangle, an altitude is a segment that joins one of the three vertices to a point on the opposite side, the intersection being perpendicular. In some triangles, it may be necessary to extend the side to meet the altitude. The *length* of this segment is also called an altitude, as is the distance that separates the parallel sides of a trapezoid. [19,34,39]



**angles** can often be identified by a single letter, but sometimes three letters are necessary. The angle shown can be called  $B$ ,  $ABC$ , or  $CBA$ . [1]

**angle of depression:** Angle formed by a horizontal ray and a line-of-sight ray that is below the horizontal. See the diagram below. [46]



**angle of elevation:** Angle formed by a horizontal ray and a line-of-sight ray that is above the horizontal. See the diagram above. [44]

**Angle-Angle-Side (corresponding):** When the parts of one triangle can be matched with the parts of another triangle, so that two pairs of corresponding angles have the same sizes, and so that one pair of corresponding sides has the same length, then the triangles are congruent. This rule of evidence is abbreviated to AAS. [12,27]

**angle bisector:** Given an angle, this ray divides the angle into two equal parts. [21]

**Angle-Bisector Theorem:** The bisector of any angle of a triangle cuts the opposite side into segments whose lengths are proportional to the sides that form the angle. [46]

## Mathematics 2 Reference

**Angle-Side-Angle:** When the parts of one triangle can be matched with the parts of another triangle, so that two pairs of corresponding angles have the same sizes, and so that the (corresponding) shared sides have the same length, then the triangles are congruent. This rule of evidence is abbreviated to ASA. [12]

**angular size of an arc:** This is the size of the central angle formed by the radii that meet the endpoints of the arc. [63]

**Apollonian circle:** A curve consisting of those points whose distances from two fixed points are in a constant ratio. [42,78] The Greek geometer Apollonius of Perga, who flourished about 2200 years ago, wrote many books, and gave the *parabola* its name.

**arc:** The portion of a circle that lies to one side of a chord is called an *arc*. [61]

**arc length:** Given a circle, the length of any arc is proportional to the size of its central angle.

**areas of similar figures:** If two figures are similar, then the ratio of their areas equals the *square* of the ratio of similarity. [58,69]

**bagel:** [16,44]

**bisect:** Divide into two pieces that are, in some sense, equal. [4,5,19,27]

**buckyball:** Named in honor of R. Buckminster Fuller, this is just another name for the *truncated icosahedron*. [21]

**central angle:** An angle formed by two radii of a circle. [63]

**centroid:** The medians of a triangle are concurrent at this point, which is the balance point (also known as the *center of gravity*) of the triangle. [23,39,41]

**chord:** A segment that joins two points on a circle is called a *chord* of the circle. [60]

**circle:** This curve consists of all points that are at a constant distance from a *center*. The common distance is the *radius* of the circle. A segment joining the center to a point on the circle is also called a *radius*. [57]

**circumcenter:** The perpendicular bisectors of the sides of a triangle are concurrent at this point, which is equidistant from the vertices of the triangle. [24]

**circumcircle:** When possible, the circle that goes through all the vertices of a polygon.

**collinear:** Three (or more) points that all lie on a single line are *collinear*. [6]

## Mathematics 2 Reference

**common chord:** The segment that joins the points where two circles intersect. [68]

**complementary:** Two angles that fit together to form a right angle are called complementary. Each angle is the *complement* of the other. [3]

**completing the square:** Applied to an equation, this is an algebraic process that is useful for finding the center and the radius of a circle, or the vertex and focus of a parabola. [74,79]

**components** describe how to move from one unspecified point to another. They are obtained by *subtracting* coordinates. [7]

**concentric:** Two figures that have the same center are called *concentric*.

**concurrent:** Three (or more) lines that go through a common point are *concurrent*. [22]

**concyclic:** Points that all lie on a single circle are called *concyclic*. [67]

**congruent:** When the points of one figure can be matched with the points of another figure, so that corresponding parts have the same size, then the figures are called *congruent*, which means that they are considered to be equivalent. [3,9,11]

**converse:** The converse of a statement of the form “if [something] then [something else]” is the statement “if [something else] then [something].” [33]

**convex:** A polygon is called *convex* if every segment joining a pair of points within it lies entirely within the polygon. [36]

**coordinates:** Numbers that describe the position of a point in relation to the origin of a coordinate system.

**corresponding:** Describes parts of figures (such as angles or segments) that have been matched by means of a transformation. [11]

**cosine ratio:** Given a right triangle, the cosine of one of the acute angles is the ratio of the length of the side *adjacent* to the angle to the length of the hypotenuse. The word cosine is a combination of *complement* and *sine*, so named because the cosine of an angle is the same as the sine of the complementary angle. [62]

**CPCTC:** *Corresponding Parts of Congruent Triangles are themselves Congruent*. [18]

**Crossed-Chords Theorem:** When two chords intersect inside a circle, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord. Thus the value of this product depends only on the location of the point of intersection. [75]

## Mathematics 2 Reference

**cyclic:** A polygon, all of whose vertices lie on the same circle, is called *cyclic*. Also called an *inscribed polygon*. [71,72]

**decagon:** A polygon that has ten sides. [42]

**diagonal:** A segment that connects two nonadjacent vertices of a polygon.

**diameter:** A chord that goes through the center of its circle is called a *diameter*. [61]

**dihedral:** An angle that is formed by two intersecting *planes*. To measure its size, choose a point that is common to both planes, then through this point draw the line in each plane that is perpendicular to their line of intersection. [63,64]

**dilation:** A similarity transformation, with the special property that all lines obtained by joining points to their images are concurrent at the same *central* point. [53,54,55]

**direction vector:** A vector that describes a line, by pointing from a point on the line to some other point on the line. [20]

**directrix:** See *parabola*.

**distance formula:** The distance from  $(x_1, y_1)$  to  $(x_2, y_2)$  is  $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ , and the distance from  $(x_1, y_1, z_1)$  to  $(x_2, y_2, z_2)$  is  $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ . These formulas are consequences of the *Pythagorean Theorem*.

**dodecagon:** A polygon that has twelve sides. [46]

**dodecahedron:** A polyhedron formed by attaching twelve polygons edge to edge. If the dodecagon is regular, each of its vertices belongs to three congruent *regular* pentagons.

**Doppler shift:** The change of frequency that results when the source of a signal is moving relative to the observer. [48]

**dot product:** Given vectors  $[a, b]$  and  $[m, n]$ , their dot product is the number  $am + bn$ . Given vectors  $[a, b, c]$  and  $[p, q, r]$ , their dot product is the number  $ap + bq + cr$ . In either case, it is the sum of the products of corresponding components. When the value is zero, the vectors are perpendicular, and conversely. [22,31]

**equiangular:** A polygon all of whose angles are the same size. [2]

**equidistant:** A shortened form of *equally distant*. [2]

**equilateral:** A polygon all of whose sides have the same length. [1]



## Mathematics 2 Reference

**Euclidean geometry** (also known as plane geometry) is characterized by its parallel postulate, which states that, *given a line, exactly one line can be drawn parallel to it through a point not on the given line*. A more familiar version of this assumption states that *the sum of the angles of a triangle is a straight angle*. [3,31] The Greek mathematician Euclid, who flourished about 2300 years ago, wrote many books, and established a firm logical foundation for geometry.

**Euler line:** The centroid, the circumcenter, and the orthocenter of any triangle are collinear. [26] The Swiss scientist Leonhard Euler (1707-1783) wrote copiously on both mathematics and physics, and knew the *Aeneid* by heart.

**exterior angle:** An angle that is formed by a side of a polygon and the extension of an adjacent side. It is supplementary to the adjacent interior angle. [32]

**Exterior-Angle Theorem:** An exterior angle of a triangle is the sum of the two nonadjacent interior angles. [32,45]

**focus:** See *parabola*.

**foot:** The point where an altitude meets the base to which it is drawn. [19,59,60]

**function:** A function is a rule that describes how the value of one thing is determined uniquely by the value of another thing. [1,5,11,12,20]

**glide-reflection:** An isometric transformation of a plane that leaves no single point fixed, but that does map a single line to itself. A glide-reflection thus maps points on either side of this line to the other side. Think of the footprints left by a person walking in a straight line. [9,21]

**Greek letters** appear often in mathematics. Some of the common ones are  $\alpha$  (alpha),  $\beta$  (beta),  $\Delta$  or  $\delta$  (delta),  $\theta$  (theta),  $\Lambda$  and  $\lambda$  (lambda),  $\mu$  (mu),  $\pi$  (pi), and  $\Omega$  or  $\omega$  (omega).

**head:** Vector terminology for the second vertex of a directed segment. [7]

**hexagon:** a polygon that has six sides. [2]

**Hypotenuse-Leg:** When the hypotenuses of two right triangles have the same length, and a leg of one triangle has the same length as a leg of the other, then the triangles are congruent. This rule of evidence is abbreviated to HL. [27]

**icosahedron:** A polyhedron formed by attaching twenty polygons edge to edge. If the polyhedron is regular, each of its vertices belongs to five equilateral triangles. [21]

**icosidodecahedron:** A polyhedron formed by attaching the edges of twenty equilateral triangles to the edges of twelve regular pentagons. Two triangles and two pentagons meet at each vertex. [27]

## Mathematics 2 Reference

**image:** The result of applying a transformation to a point  $P$  is called the *image of  $P$* , often denoted  $P'$ . One occasionally refers to an *image segment* or an *image triangle*. [20]

**incenter:** The angle bisectors of a triangle are concurrent at this point, which is equidistant from the sides of the triangle. [73]

**included angle:** The angle formed by two designated segments. [71]

**inscribed angle:** An angle formed when two chords meet at a point on the circle. An inscribed angle is *half* the angular size of the arc it intercepts. In particular, an angle that intercepts a semicircle is a *right* angle. [65]

**inscribed polygon:** A polygon whose vertices all lie on the same circle; also called a *cyclic polygon*. [71,72]

**integer:** Any whole number, whether it be positive, negative, or zero. [7]

**intercepted arc:** The part of an arc that is found inside a given angle. [65]

**isometry:** A geometric transformation that preserves distances. The best-known examples of isometries are *translations*, *rotations*, and *reflections*. [22]

**isosceles triangle:** A triangle that has two sides of the same length. [3] The word is derived from the Greek *iso* + *skelos* (equal + leg)

**Isosceles-Triangle Theorem:** If a triangle has two sides of equal length, then the angles opposite those sides are also the same size. [21]

**isosceles trapezoid:** A trapezoid whose nonparallel sides have the same length. [39]

**kite:** A quadrilateral that has two disjoint pairs of congruent adjacent sides. [2,18]

**labeling convention:** Given a polygon that has more than three vertices, place the letters around the figure in the order that they are listed. [19,20,24]

**lateral face:** Any face of a pyramid or prism that is not a base. [74]

**lattice point:** A point whose coordinates are both integers. [3]

**lattice rectangle:** A rectangle whose vertices are all lattice points. [15]

**leg:** The perpendicular sides of a right triangle are called its legs. [7]

**length of a vector:** This is the length of any segment that represents the vector. [10]

**lens:** A region enclosed by two intersecting, non-concentric circular arcs. [76]

## Mathematics 2 Reference

**linear equation:** Any straight line can be described by an equation in the form  $ax+by = c$ .

**magenta:** A shade of purple, named for a town in northern Italy. [65]

**magnitude of a dilation:** The nonnegative number obtained by dividing the length of any segment into the length of its dilated image. See *ratio of similarity*. [55]

**major/minor arc:** Two arcs are determined by a given chord. The smaller arc is called *minor*, and the larger arc is called *major*. [61]

**MasterCard:** [72]

**median of a triangle:** A segment that joins a vertex of a triangle to the midpoint of the opposite side. [20]

**median-median line:** Given a set of points, this is the *zero-residual line* determined by the three *summary points* obtained by dividing the data into three groups of equal size and applying the *median-median* procedure to each of them. [70]

**median-median point:** Given a set of points, this is the point whose  $x$ -coordinate is the median of all the given  $x$ -coordinates and whose  $y$ -coordinate is the median of all the given  $y$ -coordinates. [68]

**midline of a trapezoid:** This segment joins the midpoints of the non-parallel sides. Its length is the average of the lengths of the parallel sides, to which it is also parallel. Also known as the *median* in some books. [47]

**Midline Theorem:** A segment that joins the midpoints of two sides of a triangle is parallel to the third side, and is half as long. [36,42]

**midpoint:** The point on a segment that is equidistant from the endpoints of the segment. If the endpoints are  $(a, b)$  and  $(c, d)$ , the midpoint is  $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ . [4]

**mirror:** See *reflection*.

**negative reciprocal:** One number is the negative reciprocal of another if the product of the two numbers is  $-1$ . [3]

**octagon:** a polygon that has eight sides. [28]

**opposite:** Two numbers or vectors are opposite if they differ in sign. For example, 17.5 is the opposite of  $-17.5$ , and  $[2, -11]$  is the opposite of  $[-2, 11]$ . [10,12]

**orthocenter:** The altitudes of a triangle are concurrent at this point. [22,49]

## Mathematics 2 Reference

**parabola:** A curve consisting of those points that are equidistant from a given line and a given point form a curve called a *parabola*. The given point is called the *focus* and the given line is called the *directrix*. The point on the parabola that is closest to the directrix (thus closest to the focus) is the *vertex*. [32,44,61]

**parallel:** Coplanar lines that do not intersect. When drawn in a coordinate plane, they are found to have the same slope, or else no slope at all. The shorthand  $\parallel$  is often used.

**parallelogram:** A quadrilateral that has two pairs of parallel sides. [14]

**parameter:** See the examples on pages 5, 8, and 78.

**pentagon:** a polygon that has five sides. [25]

**perpendicular:** Coplanar lines that intersect to form a right angle. If  $m_1$  and  $m_2$  are the slopes of two lines in the  $xy$ -plane, neither line parallel to a coordinate axis, and if  $m_1 m_2 = -1$ , then the lines are perpendicular. [3]

**perpendicular bisector:** Given a line segment, this is the line that is perpendicular to the segment and that goes through its *midpoint*. The points on this line are all *equidistant* from the endpoints of the segment. [4]

**perpendicular vectors:** Two vectors whose dot product is zero.

**point-slope form:** A non-vertical straight line can be described by  $y - y_0 = m(x - x_0)$  or by  $y = m(x - x_0) + y_0$ . One of the points on the line is  $(x_0, y_0)$  and the slope is  $m$ . [5]

**postulate:** A statement that is accepted as true, without proof. [3]

**prism:** A three-dimensional figure that has two congruent and parallel *bases*, and parallelograms for its remaining *lateral faces*. If the lateral faces are all rectangles, the prism is a *right prism*. [29] If the base is a regular polygon, the prism is also called *regular*.

**probability:** A number between 0 and 1, often expressed as a percent, that expresses the likelihood that a given event will occur. For example, the probability that two coins will *both* fall showing heads is 25%.

**proportion:** An equation that expresses the equality of two *ratios*. [53]

**pyramid:** A three-dimensional figure that is obtained by joining all the points of a polygonal *base* to a *vertex*. Thus all the lateral faces of a pyramid are triangles. If the base polygon is regular, and the lateral edges are all congruent, then the pyramid is called *regular*. [74]

## Mathematics 2 Reference

**Pythagorean Theorem:** The square on the hypotenuse of a right triangle equals the sum of the squares on the legs. If  $a$  and  $b$  are the lengths of the legs of a right triangle, and if  $c$  is the length of the hypotenuse, then these lengths fit the Pythagorean equation  $a^2 + b^2 = c^2$ . [1,2] Little is known about the Greek figure Pythagoras, who flourished about 2500 years ago, except that he probably did not discover the theorem that bears his name.

**quadrant:** one of the four regions formed by the coordinate axes. Quadrant I is where both coordinates are positive, and the other quadrants are numbered (using Roman numerals) in a counterclockwise fashion.

**quadratic formula:**  $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$  and  $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$  are the two solutions to  $ax^2 + bx + c = 0$ .

**quadrilateral:** a four-sided polygon. [1,2,9,10]

**radial expansion:** See *dilation*.

**ratio of similarity:** The ratio of the lengths of any two corresponding segments of *similar* figures. [55]

**reflection:** An isometric transformation of a plane that has a line of fixed points. A reflection maps points on either side of this line (the *mirror*) to the other side. [9,21]

**reflection property of a parabola:** Through any point on a *parabola*, draw the line that is parallel to the axis of symmetry, the line that goes through the focus, and the tangent line. The first two lines make equal angles with the third. [73]

**regular:** A polygon that is both equilateral and equiangular. [25]

**regular pyramid:** See *pyramid*.

**residual:** Given a line  $y = mx + b$  and a point  $(x_1, y_1)$  not on the line, the difference  $y_1 - (mx_1 + b)$  is called a *residual*. Its magnitude is the vertical distance between the point and the line. Its sign tells whether the point is above or below the line. [57]

**Rhode Island School of Design.** [209]

**rhombus:** An equilateral quadrilateral. [1,10]

**right angle:** An angle that is its own supplement. [3,18]

**rotation:** An isometric transformation of a plane that leaves a single point fixed. [9,21]

**SAS similarity:** Two triangles are certain to be similar if two sides of one triangle are proportional to two sides of the other, and if the included angles are equal in size. [71]

## Mathematics 2 Reference

**scalar:** In the context of vectors, this is just another name for a number. [12]

**scalene:** A triangle no two of whose sides are the same length. [23]

**segment:** That part of a line that lies between two designated points. [3,4]

**Sentry Theorem:** The sum of the exterior angles (one at each vertex) of any polygon is 360 degrees. [32,35,45]

**Shared-Altitude Theorem:** If two triangles share an altitude, then the ratio of their areas is proportional to the ratio of the corresponding bases. [53]

**Shared-Base Theorem:** If two triangles share a base, then the ratio of their areas is proportional to the ratio of the corresponding altitudes. [56]

**Side-Angle-Side:** When the parts of one triangle can be matched with the parts of another triangle, so that two pairs of corresponding sides have the same lengths, and so that the (corresponding) angles they form are also the same size, then the triangles are congruent. This rule of evidence is abbreviated to just SAS. [10]

**Side-Side-Angle:** This is insufficient evidence for congruence. [11,26] See the item *Hypotenuse-Leg*, however.

**Side-Side-Side:** When the parts of one triangle can be matched with the parts of another triangle, so that all three pairs of corresponding sides have the same lengths, then the triangles are congruent. This rule of evidence is abbreviated to just SSS. [9]

**similar:** Two figures are similar if their points can be matched in such a way that all ratios of corresponding lengths are proportional to a fixed *ratio of similarity*. Corresponding angles of similar figures must be equal in size. [55]

**sine ratio:** Given a right triangle, the sine of one of the acute angles is the ratio of the length of the side *opposite* the angle to the length of the hypotenuse. [54]

**skew lines:** Non-intersecting lines whose direction vectors are not parallel. [59]

**slope:** The slope of the segment that joins the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $\frac{y_2 - y_1}{x_2 - x_1}$ .

**slope-intercept form:** Any non-vertical straight line can be described by an equation that takes the form  $y = mx + b$ . The slope of the line is  $m$ , and the  $y$ -intercept is  $b$ .

**SSS similarity:** Two triangles are *similar* if their sides are proportional. [59]

**stop sign:** [28]

## Mathematics 2 Reference

**subtended angle:** Given a point  $O$  and a figure  $\mathcal{F}$ , the *angle subtended by  $\mathcal{F}$  at  $O$*  is the smallest angle whose vertex is  $O$  and whose interior contains  $\mathcal{F}$ . [63]

**summary point:** The *median-median* point of a (sub)set of points. [70]

**supplementary:** Two angles that fit together to form a straight line are called *supplementary*. Each angle is the *supplement* of the other. [3]

**symmetry axis of a parabola:** The line through the focus that is perpendicular to the directrix. Except for the *vertex*, each point on the parabola is the reflected image of another point on the parabola. [64]

**tail:** Vector terminology for the first vertex of a directed segment. [7]

**tail-to-tail:** Vector terminology for directed segments with a common first vertex. [28]

**tangent ratio:** Given a right triangle, the tangent of one of the acute angles is the ratio of the side opposite the angle to the side adjacent to the angle. [43]

**tangent and slope:** When an angle is formed by the positive  $x$ -axis and a ray through the origin, the *tangent* of the angle is the *slope* of the ray. Angles are measured in a counterclockwise sense, so that rays in the second and fourth quadrants determine negative tangent values. [43]

**tangent to a circle:** A line that touches a circle without crossing it. Such a line is perpendicular to the radius drawn to the point of tangency. [72]

**tangent to a parabola:** A line that intersects the curve without crossing it. To draw the tangent line at a given point on a parabola, join the nearest point on the directrix to the focus, then draw the perpendicular bisector of this segment. [30,44,61]

**tessellate:** To fit non-overlapping tiles together to cover a planar region. [36,42]

**tetrahedron:** A pyramid whose four faces are all triangles. [77]

**Three-Parallels Theorem:** Given three parallel lines, the segments they intercept on one transversal are proportional to the segments they intercept on any transversal. [42,43]

**transformation:** A *function* that maps points to points. [20,23,26]

**translate:** To slide a figure by applying a vector to each of its points. [8,21]

**transversal:** A line that intersects two other lines in a diagram. [31]

**trapezoid:** A quadrilateral with exactly one pair of parallel sides. If the non-parallel sides have the same length, the trapezoid is called *isosceles*. [39]

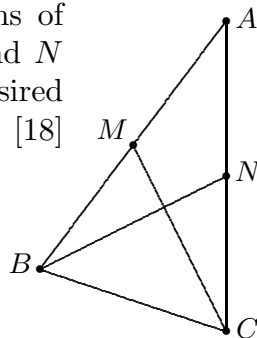
## Mathematics 2 Reference

**triangle inequality:** For any  $P$ ,  $Q$ , and  $R$ ,  $PQ \leq PR + RQ$ . It says that any side of a triangle is less than or equal to the sum of the other two sides. [13,40]

**truncated icosahedron:** A polyhedron obtained by slicing off the vertices of an icosahedron. The twelve icosahedral vertices are replaced by twelve pentagons, and the twenty icosahedral triangles become twenty hexagons. [21]

**two-column proof:** A way of outlining a geometric deduction. Steps are in the left column, and supporting reasons are in the right column. For example, here is how one might show that an isosceles triangle  $ABC$  has two medians of the same length. It is given that  $AB = AC$  and that  $M$  and  $N$  are the midpoints of sides  $AB$  and  $AC$ , respectively. The desired conclusion is that medians  $CM$  and  $BN$  have the same length. [18]

$AB = AC$	given
$AM = AN$	$M$ and $N$ are midpoints
$\angle MAC = \angle NAB$	shared angle
$\triangle MAC \cong \triangle NAB$	SAS
$CM = BN$	CPCTC



**Two-Tangent Theorem:** From a point outside a circle, there are two segments that can be drawn tangent to the circle. These segments have the same length. [75]

**unit circle:** This circle consists of all points that are 1 unit from the origin  $O$  of the  $xy$ -plane. Given a point  $P$  on this circle, the coordinates of  $P$  are the *cosine* and the *sine* of the counterclockwise angle formed by segment  $OP$  and the positive  $x$ -axis. [65]

**unit square:** Its vertices are  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ . [28]

**Varignon parallelogram:** Given any quadrilateral, this is the figure formed by connecting the midpoints of consecutive sides. [49] The French mathematician Pierre Varignon (1654-1722) learned calculus when it was a new science, then taught it to others.

**vectors** have *magnitude* (size) and *direction*. Visualize them as directed segments (arrows). Vectors are described by *components*, just as points are described by coordinates. The vector from point  $A$  to point  $B$  is often denoted  $\overrightarrow{AB}$ , or abbreviated by a boldface letter such as  $\mathbf{u}$ , and its magnitude is often denoted  $|\overrightarrow{AB}|$  or  $|\mathbf{u}|$ . [7,12,30]

**vector triangles:** Given vectors  $\mathbf{u} = [a, b]$  and  $\mathbf{v} = [c, d]$ , a triangle is determined by drawing  $\mathbf{u}$  and  $\mathbf{v}$  so that they have a common initial point (*tail-to-tail*). No matter what this initial point is, the triangles determined by  $\mathbf{u}$  and  $\mathbf{v}$  are all congruent. All have  $\frac{1}{2}|ad - bc|$  as their area. [25]

**velocity:** A vector obtained by dividing a displacement vector by the elapsed time.



## Mathematics 2 Reference

**vertex:** A labeled point in a figure. The plural is *vertices*, but “vertice” is not a word. The point on a parabola that is closest to the focus is also called the vertex. [64]

**vertical angles:** Puzzling terminology that is often used to describe a pair of nonadjacent angles formed by two intersecting lines. [14]

**volume of a prism:** This is the product of the *base area* and the *height*, which is the distance between the parallel base planes.

**volume of a pyramid:** This is one third of the product of the *base area* and the *height*, which is the distance from the vertex to the base plane.

**volumes of similar figures:** If two three-dimensional figures are similar, then the ratio of their volumes equals the *cube* of the ratio of similarity.

**zero-residual line:** If it is suspected that three data points should conform to a linear model, one possibility is the *zero-residual line*. This line is parallel to the line through the leftmost and rightmost points, and it makes the sum of the three residuals zero. Unless the three points are collinear, none of the three points is actually *on* the zero-residual line. [60,62]